ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Macroeconomics xxx (2005) xxx–xxx
Journal of MACROECONOMICS
www.elsevier.com/locate/jmacro

Discussion

Testing for nonlinear structure and chaos in economic time series: A comment

Cars H. Hommes *, Sebastiano Manzan

Department of Economics, CeNDEF, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, Netherlands

Received 27 September 2005; accepted 24 October 2005

Abstract

This short paper is a comment on "Testing for Nonlinear Structure and Chaos in Economic Time Series" by Catherine Kyrtsou and Apostolos Serletis. We summarize their main results and discuss some of their conclusions concerning the role of outliers and noisy chaos. In particular, we include some new simulations to investigate whether economic time series may be characterized by low-dimensional *noisy* chaos.

JEL classification: C22; C45; C61

Keywords: Nonlinearity; Chaos; Self-organized criticality; Mackey–Glass–GARCH

1. Introduction

Since the mid eighties several economists have tried to test for nonlinearity and in particular for chaos in economic and financial time series (e.g. Brock and Sayers, 1988; Scheinkman and LeBaron, 1989). In order to test for chaos, two quantities may be derived from a time series. Firstly, one can estimate the correlation dimension measuring the fractal nature of a possibly underlying strange attractor. Secondly, one can estimate the largest

* Corresponding author. Tel.: +31 20 525 4246; fax: +31 20 525 4349.
E-mail address: c.h.hommes@uva.nl (C.H. Hommes).

0164-0704/$ - see front matter
doi:10.1016/j.jmacro.2005.10.012

Lyapunov exponent which, when found to be positive, measures the sensitive dependence on initial conditions so characteristic of a chaotic system.

The methods to detect chaos however are highly sensitive to noise (see e.g. Barnett and Serletis (2000) for an extensive discussion). In particular, estimation of the correlation dimension turned out to be difficult for economic and financial time series. Brock et al. (1996) developed a statistical test for independence, known as the BDS-test, based on the correlation integral, which can be used as a general specification test. More recently, an important step forward has been made by Shintani and Linton (2004), who derived the asymptotic distribution of a nonparametric neural network estimator of the Lyapunov exponent of a noisy system. Since one frequently used definition of chaos is a positive largest Lyapunov exponent, this test may be seen as a direct test for chaos. Recently, this method has been applied by Shintani and Linton (2003) to test for chaos in real output series from various countries. For most series they find a statistically negative Lyapunov exponent, thus rejecting the hypothesis of chaos.

The current special issue of the *Journal of Macroeconomics* contains several papers on testing for nonlinearity and chaos. In this comment we discuss the paper *Univariate Tests for Nonlinear Structure*, by Kyrtsou and Serletis (this issue). After a discussion of the various nonlinearity tests in the paper, we discuss the implications of their findings, in particular concerning the question: *are economic time series characterized by low-dimensional noisy chaos*? This question has generated some controversy in the last 15 years. For example, Granger (1991, 1994) has written critical reviews on modeling economic phenomena by deterministic chaotic models. In order to shed some new light on this important issue we apply the recently developed methods of Shintani and Linton (2004) to a simple low-dimensional chaotic stock market model of Brock and Hommes (1998) buffeted with dynamic noise, to check the robustness of a positive estimate of the Lyapunov exponent.

2. Results

Kyrtsou and Serletis (this issue) use a set of 10 tests that have power to detect nonlinearities of various types. Nonlinearity can occur in the first moment of the process as well as in the conditional variance (GARCH-type dynamics) or even higher moments. In addition, they consider the raw series of daily returns of the USD/CAD exchange rate as well as a filtered series where outliers are removed. The tests suggest the following conclusions:

- *Linearity in mean*: the White neural network test strongly rejects linearity for the unfiltered exchange rate series, but does not reject linearity when the outliers are removed. Inference using the Theiler surrogate data approach leads to the same conclusions. The Bicovariance, Bispectral and Tsay statistics can also be considered as tests for linearity of the conditional mean. They evaluate the significance of cross-products of lagged values of the time series. Another way of thinking about these tests is that they test for linearity of the third conditional moment of the process, the skewness. These tests reject (for both raw and filtered returns) the null hypothesis of linearity, although for the Tsay test only at the 10% significance level. Hence, it can be concluded that for the returns of the USD/CAD exchange rate there is evidence for nonlinear dependence between the time series and interaction terms of lagged values. Also, it can be interpreted as evidence that the dependence occurs in the third conditional moment rather than in the first.

- *Heteroskedasticity*: another form of nonlinear dependence occurs when the conditional variance is time-varying. The authors consider the McLeod-Li and Engle tests for dependence in the conditional variance. The results strongly suggests the rejection of the null of a constant second moment. This is largely in accordance with the widely accepted GARCH effect in economic and financial time series. In addition, the results do not dependend on the filtering procedure for outliers.
- *General dependence test*: the BDS-test is a general test for dependence. Rejections occur when the process has dependence in any moment of the distribution. In all cases, the BDS-test rejects the hypothesis of IID observations. The results are thus consistent with the above evidence of structure in the second and third conditional moment.
- *Chaos*: the Lyapunov exponent (LE) test for low-dimensional chaos clearly suggest that for both the raw exchange rate return series as well as the filtered series the LE is significantly negative. This indicates that the series is consistent with a stochastic process rather than a deterministic low-dimensional chaotic system. The authors note however that the results may still be consistent with high-dimensional (noisy) chaos. In another paper by Serletis and Shintani (this issue) in this special issue similar results, i.e. a statistically significant negative Lyapunov exponent, are found for monetary time series of Canadian and US simple-sum Divisa and currency equivalent money and velocity measures.

3. New challenges

The paper of Kyrtsou and Serletis (this issue) also contributes in reviving two long-standing debates in the nonlinear economics dynamics community. The first relates to the role played by outliers in testing for nonlinearity. The second is associated with the interpretation of the results to test for chaos. We will now discuss these two issues in some more detail.

3.1. Outliers: Exogenous or endogenous?

This issue relates to the interpretation of extreme observations: are they the results of large exogenous shocks or are they inherently related to the dynamical behavior of the model? In other words, are they exogenous phenomena that we better neglect in empirical work or are they caused by strong nonlinearities? This is a very important issue, for example if we are interested in forecasting extreme events. The exogenous view suggests that extreme events are unpredictable and simply neglects them. The nonlinear dynamics approach views them as endogenous to the system and is informative about their generating mechanism.

Further evidence on the relevance of the issue is provided by the authors in Section 6. They estimate a model with a nonlinear structure in the conditional mean, the Generalized Mackey–Glass (GMG) model (motivated by and related to the high-dimensional chaotic Mackey–Glass system) together with a GARCH-model for the conditional variance. They found that for the raw returns there is strong evidence in support of the proposed model. However, when the outliers are removed the best performing model is a simple GARCH(1, 1) model.

3.2. Is the economy characterized by low-dimensional noisy chaos?

Application of the LE-test to economic time series suggests that there is no evidence to support the positivity of the exponent and thus that we are dealing with a stochastic system. Most experts note that the null of high-dimensional chaos has not been rejected, because it is extremely difficult to distinguish between high-dimensional chaos and randomness and one would need extremely long time series to do so. Moreover, the tests are highly sensitive to noise and this becomes worse when the dimension of the system increases. But has low-dimensional noisy chaos been rejected as a null? It is remarkable that this important question has not received much attention in the literature. The main reason seems to be that well known chaotic maps such as the one-dimensional quadratic logistic map and the two-dimensional quadratic Hénon-map, only allow for extremely small levels of dynamic noise, because small noise easily causes the system to diverge to infinity in the chaotic parameter range.

In order to shed some light on this important issue, we consider as an example the chaotic asset pricing model with heterogeneous beliefs proposed by Brock and Hommes (1998) buffeted with dynamic noise. For suitable parameters in the chaotic region, we can push the dynamic noise level to large values while keeping the system bounded, and we can thus investigate how far we can push the noise level before the positive Lyapunov exponent of the underlying chaotic skeleton model becomes negative due to the presence of dynamic noise. The model assumes that agents hold different beliefs about the future asset price and switch endogenously between the different beliefs types based on their past performance. The nonlinear dynamic model is

$$x_t = \frac{1}{R}\sum_{h=1}^{4} n_{h,t}(g_h x_{t-1} + b_h) + \sigma\epsilon_t, \tag{1}$$

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{\sum_{j=1}^{4} e^{\beta U_{j,t-1}}}, \tag{2}$$

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2})(g_h x_{t-3} + b_h - Rx_{t-2}). \tag{3}$$

Here x_t denotes the deviation of price of the risky asset from its benchmark fundamental value (the discounted sum of expected future dividends), $R > 1$ is the constant gross risk free rate, $n_{h,t}$ represents the discrete choice fraction of agents using belief type h, $U_{h,t-1}$ is the profit generated by strategy h in the previous period, g_h and b_h characterize the linear belief with one time lag of strategy h, and the noise term ϵ_t is standard normally distributed with σ the standard deviation of the dynamic noise component. Brock and Hommes (1998) show that for suitable choices of the parameter values (especially when the intensity of choice β to switch strategies is high) the 4-type version of the deterministic skeleton of the model exhibits complicated, chaotic dynamics. The stochastic version of the model (1) adds dynamic noise to the deterministic structure. Notice that substituting Eqs. (2) and (3) into (1), the model is in fact a nonlinear difference equation with three lags, i.e. it is of the form

$$x_t = F(x_{t-1}, x_{t-2}, x_{t-3}) + \sigma\epsilon_t, \tag{4}$$

which is equivalent to a three-dimensional nonlinear first order system. An advantage of the model is that, for suitable choices of the parameters in the chaotic region, it does not

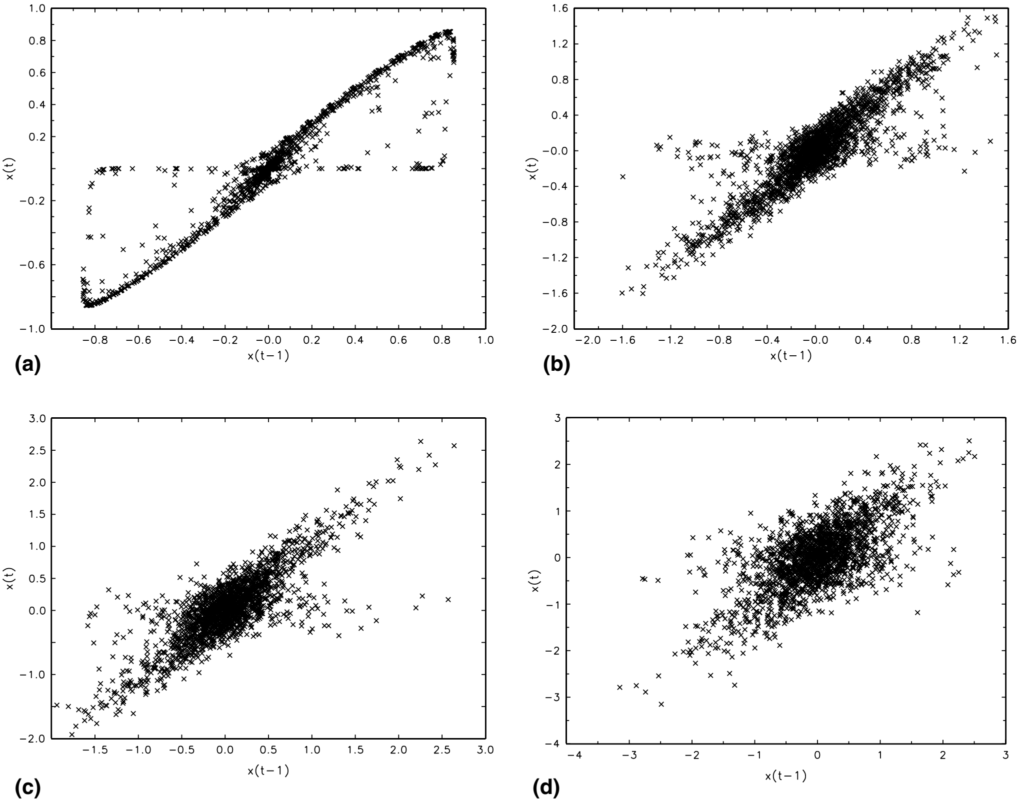

Fig. 1. Delay plots (x_{t-1}, x_t) of the nonlinear model (1) for the deterministic case and for three different noise levels; (a) shows (a projection of) the strange attractor of the deterministic skeleton. Parameters are: $R = 1.01$, $\beta = 90$, $g_1 = b_1 = 0$, $g_2 = 0.9$, $b_2 = 0.2$, $g_3 = 0.9$, $b_3 = -0.2$, $g_4 = 1.01$ and $b_4 = 0$. (a) Deterministic: $\sigma = 0$, (b) $\sigma = 0.1$, (c) $\sigma = 0.2$ and (d) $\sigma = 0.4$.

explode when the noise interacts with the deterministic dynamics. Fig. 1 shows the attractors of time series from the deterministic case and for different noise levels σ.

We now apply the LE-test[1] to a time series (2000 observations) generated by the model in the deterministic and stochastic case. This exercise is only for illustrative purposes and a detailed analysis of the behavior of the LE for the stochastic system would require Monte Carlo simulations. We used three lags in the estimation of the neural network (corresponding to the true dimension 3 of the system) and four hidden units (corresponding to a sum of four sigmoid functions in Eq. (1) and similar to values used in empirical applications). The results are shown in Table 1. For the deterministic case we find that the LE is significantly positive with an estimated value $\lambda \approx 0.135$ close to the value $\lambda \approx 0.12$ obtained with the direct method for estimating the LE of Wolff et al. (1985). However, when we increase the noise level, the estimated LE becomes smaller and even negative. Only for the smallest noise level $\sigma = 0.05$ we obtain a slightly, but significantly, positive LE $\lambda \approx 0.038$. For $\sigma = 0.1$ the estimated LE is very close to 0 (slightly negative, but not significant). For

[1] We would like to thank Mototsugu Shintani for kindly providing his programs to compute the LE-statistic.

Table 1
LE estimates (with t-statistics in parenthesis) of the neural network model with three lags and four hidden units for time series of 2000 observations for various noise levels σ

	SN	LE-(3,4)
Deterministic	0	0.135 (13.6)
$\sigma = 0.05$	0.12	0.038 (3.53)
$\sigma = 0.1$	0.22	−0.003 (−0.313)
$\sigma = 0.2$	0.36	−0.028 (−2.24)
$\sigma = 0.3$	0.48	−0.057 (−5.44)
$\sigma = 0.4$	0.55	−0.07 (−5.79)

SN is the (inverse) signal-to-noise ratio defined as $\sigma/\sqrt{\mathrm{var}(x_t)}$.

$\sigma = 0.2$ we find a statistically significant negative estimated LE $\lambda \approx -0.028$. In terms of the inverse signal-to-noise (SN) ratios, measured as $\mathrm{SN} = \sigma/\sqrt{\mathrm{var}(x_t)}$, $\sigma = 0.1$ corresponds to $\mathrm{SN} = 0.22$ and $\sigma = 0.2$ corresponds to $\mathrm{SN} = 0.36$. This evidence suggests that finding a negative exponent does not imply that low-dimensional noisy chaos has been rejected. In the presence of a relatively small amount of dynamic noise a chaotic model may have a negative LE although the deterministic skeleton is chaotic.

4. Conclusion

Several papers in this special issue show that the evidence for nonlinearity is strong. It is not clear which nonlinear model offers the best explanation for this detected structure, and this remains an important topic for future work. Our simulations show that a fairly small amount of dynamic noise may lead to a negative LE estimate for a noisy chaotic system. This suggests that low-dimensional chaos may still explain a significant part of observed fluctuations in economic and financial time series.

References

Barnett, W.A., Serletis, A., 2000. Martingales, nonlinearity and chaos. Journal of Economic Dynamics and Control 24, 703–724.
Brock, W.A., Hommes, C.H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control 22, 1235–1274.
Brock, W.A., Sayers, C.L., 1988. Is the business cycle characterized by deterministic chaos? Journal of Monetary Economics 22, 71–90.
Brock, W.A., Dechert, W.D., Scheinkman, J.A., LeBaron, B., 1996. A test for independence based on the correlation dimension. Econometric Reviews 15, 197–235.
Granger, C.W.J., 1991. Developments in the nonlinear analysis of economic series. Scandinavian Journal of Economics 93, 263–276.
Granger, C.W.J., 1994. Is chaotic economic theory relevant for economics? A review article of: Jess Benhabib: Cycles and chaos in economic equilibrium. Journal of International and Comparative Economics 3, 139–145.
Kyrtsou, C., Serletis, A., this issue. Univariate tests for nonlinear structure. Journal of Macroeconomics.
Scheinkman, J., LeBaron, B., 1989. Nonlinear dynamics and stock returns. Journal of Business 62, 311–337.
Serletis, A., Shintani, M., this issue. Chaotic monetary dynamics with confidence. Journal of Macroeconomics.
Shintani, M., Linton, O., 2003. Is there chaos in the world economy? A nonparametric test using consistent standard errors. International Economic Review 44, 331–358.
Shintani, M., Linton, O., 2004. Nonparametric neural network estimation of Lyapunov exponents and a direct test for chaos. Journal of Econometrics 120, 1–33.
Wolff, A., Swift, J.B., Swinney, H.L., Vastano, J.A., 1985. Determining Lyapunov exponents from a time series. Physica D 16, 285–317.