# Forecasting the Return Distribution Using High-Frequency Volatility Measures

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**Abstract**

The aim of this paper is to forecast (out-of-sample) the distribution of financial returns based on realized volatility measures constructed from high-frequency returns. We adopt a semi-parametric model for the distribution by assuming that the return quantiles depend on the realized measures and evaluate the distribution, quantile and interval forecasts of the quantile model in comparison to a benchmark GARCH model. The results suggest that the model outperforms an asymmetric GARCH specification when applied to the S&P 500 futures returns, in particular on the right tail of the distribution. However, the model provides similar accuracy to a GARCH(1,1) model when the 30-year Treasury bond futures return is considered.

*JEL Classification*: C14; C22; C53

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# 1 Introduction

Until recently, the predominant approach in modeling the conditional distribution of returns was represented by the ARCH-GARCH model proposed by Engle (1982) and Bollerslev (1986) and followed by a myriad of sophisticated refinements to the baseline model. The GARCH model introduces time variation in the conditional distribution largely through the conditional variance, and has been successful in explaining several empirical features of asset returns, such as fat tails and the slowly decaying autocorrelation in squared returns. While the GARCH model assumes a parametric form for the latent variance of returns, the recent availability of high-frequency data has sparked a growing literature of volatility estimators that do not require researchers to specify a model. The so-called realized volatility literature (see Andersen and Bollerslev, 1998, Andersen *et al.*, 2001a and 2001b among others) uses high-frequency data to proxy for the volatility of lower frequency returns, for instance, summing intra-day squared returns to estimate the daily variance. In this way, the latent variance process is observable and measured by realized volatility which facilitates the task of modeling and forecasting using time series models. Several recent papers incorporate these measures within a parametric volatility model for the dynamics of daily returns (see Shephard and Sheppard, 2010, Brownlees and Gallo, 2010, Maheu and McCurdy, 2011, and Hansen *et al.*, 2011).

In this paper we propose to relate the realized volatility measures and returns by assuming that these measures represent the driving force for the variation of the quantiles of the cumulative multi-period return distribution. In particular, the flexibility of the quantile regression model (see Koenker and Bassett, 1978) allows to consider several specifications that include smoothed versions of the realized volatility measures, the return standardized by realized volatility and nonlinear transformations of the return that are considered to account for the leverage effect. The fact that the parameters of the quantile regression model are specific to each quantile level allows the variables to have heterogeneous effects in different parts of the return distribution. In addition, the quantile model does not require to specify a distribution for the error as it is instead the case for GARCH models or for models based on realized volatility measures. Hence, the flexibility in choosing the most appropriate explanatory variables, the adaptability of the effect of these variables at each quantile level, and the distribution-free character of the method are the three characteristics that distinguish our approach from the models recently proposed in the literature that relate realized volatility and returns. The application of quantile regression to modeling and forecasting financial returns has experienced a recent surge of interest due to the emergence of risk management and its focus on forecasting the return quantiles (see Engle and Manganelli, 2004, Xiao and Koenker, 2009, Zikes, 2010, and Gaglianone *et al.*, 2011). Another aspect that distinguishes our paper is the method adopted to evaluate the performance of quantile and distribution forecasts. A common loss function used in the comparison of density forecasts is the logarithmic score rule, which rewards forecasts that have higher density at the realization of the variable being forecast (see, among others, Bao *et al.*, 2007, Amisano and Giacomini, 2007, and Maheu and McCurdy, 2011, and Shephard and Sheppard, 2010 for two applications in the realized volatility literature). Although this is certainly a relevant criterion to consider, it does not reward forecasts that assign high probabilities to values close to the realization in addition to the fact that it cannot be easily adapted to evaluate specific areas of the distribution, for instance the left or right tail. Gneiting and Raftery (2007) and Gneiting and Ranjan (2011) discuss alternative rules that overcome these problems, and we consider several of these rules to evaluate different characteristics of the return distribution. In particular, we use the Quantile Score rule represented by the tick loss function which is targeted to quantile forecasts, such as VaR (e.g. Clements *et al.*, 2008). Instead of focusing on few quantiles of interests, we examine several of them that span the complete return distribution which allows to evaluate the forecast performance of the competing models in different areas of the distribution. Furthermore, we also consider a weighted version of the Quantile Score rule that evaluates specific areas of the forecast distribution, for instance the left and right tail or the center of the distribution, and a scoring rule that evaluates interval forecasts at the 50 and 90% level.

In the empirical application we consider the S&P 500 index futures (SP) and the 30-year Treasury bond futures (US) and forecast out-of-sample the cumulative return at the 1, 2, and 5-day horizons. We evaluate and compare the forecasts from the realized volatility quantile model to those of a benchmark GARCH model represented by the GJR specification (Glosten *et al.*, 1993) for the SP returns and the simple GARCH(1,1) for the US returns. We consider several high-frequency measures of volatility that have been proposed in the literature, including several adjustments that account for the presence of microstructure noise and jumps. The results for the SP returns indicate that the distribution forecasts at the 1-day horizon from the realized volatility models outperform those from the GJR model, with the improved performance mostly deriving from the better ability to forecast the right tail of the return distribution. Only the specifications that include an asymmetric effect are able to beat GJR in modeling the left tail, and significantly so for quantile levels between 20 and 30%. Furthermore, the comparison suggests that the realized measures of volatility considered deliver very similar results, thus indicating that filtering out the effect of jumps and micro-structure noise does not improve the (out-of-sample) forecasting ability in any part of the return distribution. In addition, we also consider some quantile specifications that use absolute daily returns (and their transformations), instead of the realized measures, and the evidence indicates that their forecasts do not outperform those from the GJR benchmark. This result indicates that the flexibility of the quantile model combined with the (absolute) returns produces forecasts that have similar accuracy relative to GARCH models, although it does not require to assume a parametric specification. In addition, the realized volatility measures provide valuable information that can be used to improve the accuracy of forecasts relative to (quantile or GARCH) models that only use returns. However, the evidence for the US bond return shows that the realized volatility quantile models provide similarly accurate forecasts relative to those of the benchmark GARCH(1,1) model at all horizons. In this case thus the realized volatility measures do not provide additional forecasting power for the return distribution compared to what is already embedded in daily returns, contrary to the results for the equity index returns.

The paper is organized in this manner. Section (2) describes the realized measures of volatility that are considered in this paper, while Section (3) introduces the GARCH specifications and the semi-parametric model that we propose to incorporate the realized measures in modeling return quantiles. Section (4) describes the forecast evaluation methods and Section (5) reports the results of the empirical application. Finally, Section (7) concludes.

## 2 Realized volatility estimators

The availability of high-frequency data has sparked the development of methods to estimate the (latent) volatility of financial returns that do not require the specification of a model. The most well-known quantity is realized volatility which is obtained by summing intra-day squared returns and can be used to proxy for integrated volatility (see Andersen and Bollerslev, 1998, Barndorff-Nielsen and Shephard, 2002a, 2002b, Meddahi, 2002). In this Section, we present several realized volatility measures that are later used in our empirical application.

Denote the intra-day return in day $t$ by $r_{t,i} = ln(P_{t_i}) - ln(P_{t_{i-1}})$, where $i = 1, 2, \ldots, m$ indicates the intra-day interval and $P_{t_i}$ the asset price in interval $i$ of day $t$. The realized volatility estimator in day $t$, denoted by $RV_t$, represents a model-free estimator of the daily quadratic variation at sampling frequency $m$ and is given by

$$RV_t = \sum_{i=1}^{m} r_{t,i}^2. \tag{1}$$

The asymptotic distribution of $RV_t$ has been studied by Andersen and Bollerslev (1998), Andersen *et al.* (2001b), Andersen *et al.* (2003), and Barndorff-Nielsen and Shephard (2002a), among others. An important role in the construction of the measure is played by the selection of the sampling frequency $m$ which is complicated by several market microstructure issues (see, e.g. Ait-Sahalia *et al.*, 2005a, Ait-Sahalia *et al.*, 2005b, Bandi and Russell, 2008, Hansen and Lunde, 2006b, and Barndorff-Nielsen *et al.*, 2008, among others). In our empirical application, we use a five-minute sampling frequency, which has been shown in the literature to strike a reasonable balance between the desire for as finely sampled observations as possible and robustness to market microstructure contaminations.

Despite the careful selection of sampling frequency, market-microstructure dynamics could still cause $RV_t$ to be a biased and inconsistent estimator of volatility. Thus, we also consider estimators with adjustments that reduce market microstructure frictions present in high-frequency returns. We adopt a kernel-based estimator of realized volatility suggested by Hansen and Lunde (2006b), which employs Bartlett weights,

$$RV(q)_t = \sum_{i=1}^{m} r_{t,i}^2 + 2\sum_{w=1}^{q}\left(1 - \frac{w}{q+1}\right)\sum_{i=1}^{m-w} r_{t,i} r_{t,i+w}, \tag{2}$$

where $r_{t,i}$ is defined as above and $\left(1 - \frac{w}{q+1}\right)$ represents the weight that follows a Bartlett scheme. This estimator utilizes higher-order auto-covariances to eliminate the bias of $RV_t$, and is also guaranteed to be non-negative. The asymptotic properties of the estimator are discussed by Barndorff-Nielsen *et al.* (2008).

Volatility can also experience frequent jumps. Andersen *et al.* (2007) suggest that most of the predictable variation in the volatility stems from the continuous price path variability while the predictability of the jump component of volatility is typically minor. However, Wright and Zhou (2009) find that measures of realized jumps are useful predictors for bond risk premia. We follow the approach proposed by Andersen *et al.* (2011) to decompose the total return variability over the trading day into its continuous sample path variation and the variation due to jumps. First, we rely on the realized bipower variation measure developed by Barndorff-Nielsen and Shephard (2004),

$$RBV_t = \delta_1^{-2} \left( \frac{m}{m-2} \right) \sum_{i=3}^{m} |r_{t,i}||r_{t,i-2}|, \tag{3}$$

where $\delta_1 = \pi/2$, and more generally $\delta_\eta = E(|Z|^\eta)$ for $Z \sim N(0,1)$. The bipower measure helps render certain types of market microstructure noise as shown by Huang and Tauchen (2005). To detect a jump, we then define the jump detection test statistics as:

$$Z_t = \frac{\frac{RV_t - RBV_t}{RV_t}}{\sqrt{\left( \left(\frac{\pi}{2}\right)^2 + \pi - 5 \right) \frac{1}{m} max\left(1, \frac{RTQ_t}{RBV_t^2}\right)}}, \tag{4}$$

where

$$RTQ_t = m \left(\delta_{4/3}\right)^{-3} \left( \frac{m}{m-4} \right) \sum_{i=5}^{m} |r_{t,i}|^{4/3} |r_{t,i-2}|^{4/3} |r_{t,i-4}|^{4/3}. \tag{5}$$

Based on the jump detection test statistics, the realized measure of the jump contribution to the quadratic variation is then:

$$J_t = I(Z_t > \phi_\lambda) \left( RV_t - RBV_t \right), \tag{6}$$

where $I$ is the indicator function and $\phi_\lambda$ refers to an appropriate critical value from the standard normal distribution (in the empirical application we use the 99% critical level) and the continuous component of the realized measure is defined by

$$C_t = I(Z_t > \phi_\lambda) RBV_t + I(Z_t \leq \phi_\lambda) RV_t. \tag{7}$$

Some markets, e.g. equity markets, are closed for a part of each 24-hour period, and the opening price on one day typically differs from the closing price on the previous day. For investors that hold their portfolios over long horizons (i.e. multiple days and beyond), the overnight return variability can directly affect their positions. Moreover, Hansen and Lunde (2006a) argue that the volatility estimator that ignores the overnight period (i.e., the market is closed) might not be a proper proxy for the true volatility. To account for the overnight return, defined as $r_{t,0} = \ln(P_{t,0}) - \ln(P_{t-1,m})$, we adopt the solution proposed by Blair *et al.* (2001) which consists of adding the squared overnight return as one of the factors in the sum of intra-day returns, that is,

$$RVN_t = r_{t,0}^2 + RV_t = \sum_{i=0}^{m} r_{t,i}^2. \tag{8}$$

In the empirical application, we consider these realized volatility estimators in order to evaluate their informational content to forecast the distribution of financial returns.

# 3 Forecasting models

As discussed earlier, the aim of the analysis is to evaluate the relevance of incorporating high-frequency measures of volatility to forecast (out-of-sample) the return distribution as opposed to adopting a GARCH-type time-series model. In this Section, we first introduce the GARCH specifications from which we select the benchmark for the forecast comparison, followed by a discussion of the approach we propose to incorporate high-frequency volatility measures in the return distribution.

## 3.1 GARCH models

Hansen and Lunde (2005) provide an extensive comparison of the (out-of-sample) volatility forecasts accuracy of 330 GARCH-type specifications. Their results show that the simple GARCH(1,1) model of Bollerslev (1986) is hardly outperformed by more sophisticated specifications when forecasting daily exchange rate volatility, although it is beaten by models that include a leverage effect when forecasting stock returns. Based on these results, we decided to consider in our out-of-sample comparison a small set of models which comprise the simple GARCH(1,1) and two of the most frequently used specifications with leverage effects, the EGARCH of Nelson (1991) and the GJR-GARCH model of Glosten *et al.* (1993).

We denote the close-to-close return on day $t+1$ by $r_{t+1} = \ln(P_{t+1}) - \ln(P_t)$, where $P_t$ indicates the closing price of the asset in day $t$. We assume that the returns follow a time-varying location-scale model given by $r_{t+1} = \mu + \epsilon_{t+1}$, where $\mu$ is a constant and $\epsilon_{t+1} = \sigma_{t+1}\eta_{t+1}$, with $\sigma_{t+1}$ denoting the conditional standard deviation in day $t+1$ and $\eta_{t+1}$ is an i.i.d. error term with mean zero and variance one. To fully specify the distribution of the return $r_{t+1}$ we need to introduce assumptions on both the dynamics of the conditional variance and the distribution of the error term $\eta_{t+1}$. As concerns the conditional variance, we consider three specifications:

1. *GARCH(1,1)*: $\sigma^2_{t+1} = \omega + \alpha\epsilon^2_t + \beta\sigma^2_t$
2. *GJR-GARCH(1,1)*: $\sigma^2_{t+1} = \omega + \alpha\epsilon^2_t + \gamma\epsilon^2_t I(\epsilon_t < 0) + \beta\sigma^2_t$
3. *EGARCH(1,1)*: $\ln\left(\sigma^2_{t+1}\right) = \omega + \alpha(|\eta_t| - E|\eta_t|) + \gamma\eta_t + \beta\ln\left(\sigma^2_t\right)$

where $\omega, \alpha, \beta$ and $\gamma$ are parameters. The characteristic of the GJR and EGARCH specifications is to allow the current shock to have an asymmetric effect on volatility. The empirical evidence suggests that volatility increases more following negative surprises compared to positive ones of the same magnitude. The second hypothesis concerns the distribution of the error term $\eta_{t+1}$. In this case, we consider the parametric assumptions that $\eta_{t+1}$ is distributed according to the standard normal distribution, or that it follows the Student $t_k$ distribution, where $k$ indicates the degrees of freedom. In both cases we estimate the parameters by Maximum Likelihood. In addition to these two parametric assumptions, we also consider a non-parametric approach which consists of resampling from the Empirical Distribution Function (EDF) of the standardized residuals. The advantage of using this approach is that we abstract from assuming a parametric form, and instead let the data indicate the shape of the distribution that might, possibly, be characterized by (unconditional) skewness and excess kurtosis (see Bali, 2003, Bali and Theodossiou, 2007, and Bali and Weinbaum, 2007, for the application of Extreme Value Theory and skewed fat-tailed distribution to forecast

Value-at-Risk and Bali *et al.*, 2008, for an approach that allows for time-varying skewness and kurtosis). The combination of a GARCH-type specification for the conditional variance and errors resampled from the EDF is typically referred to as Filtered Historical Simulation (FHS). Barone-Adesi *et al.* (2011) provide a recent application of FHS to option pricing.

In addition to the one-day ahead return distribution, we are also interested in forecasting the cumulative $h$-day ahead return defined as $r^h_{t+h} = \ln(P_{t+h}) - \ln(P_t)$. The conditional variance of the $h$-day cumulative return for the GARCH(1,1) and GJR models is obtained by $\sigma^2_{t+h} = \sum_{j=1}^{h} \sigma^2_{t+j}$, which is then combined with the two parametric assumptions on the error distribution to obtain the $h$-day ahead forecast of the cumulative return CDF, denoted by $F_{t+h}(\cdot)$, and the predictive quantile at level $\tau$ by $F^{-1}_{t+h}(\tau)$. Instead, for the FHS approach the distribution of $r^h_{t+h}$ is approximated by simulating a large number $B$ of return paths in two steps: (1) estimate the GARCH(1,1), GJR, or EGARCH specification by quasi-ML and obtain the standardized residuals, and (2) iterate forward the model using the estimated parameters and the resampled standardized residuals as innovations. This provides $B$ return paths based on the one-step ahead model discussed above and the distribution of the cumulative return can be approximated by the EDF of $r^h_{t+h,b} = \sum_{j=1}^{h} r_{t+j,b}$, where $b = 1, \cdots, B$.

## 3.2 Realized volatility models

As discussed earlier, realized volatility measures represent *model-free* estimates of volatility, in the sense that they do not rely on the assumption of a parametric model, for instance a GARCH model. Andersen *et al.* (2001a, 2001b, 2003), among others, propose time series models for realized volatility measures (or a log-transformation) in order to explain their in-sample dynamics as well as to forecast and evaluate the volatility process. Recently, several papers (e.g. Engle and Gallo, 2006, Bollerslev *et al.*, 2009, Shephard and Sheppard, 2010, Brownlees and Gallo, 2010, Maheu and McCurdy, 2011, and Hansen *et al.*, 2011) depart from the univariate analysis of realized volatility and propose joint models of the dynamics of returns and realized volatility measures.

In this paper, we use the realized measures to explain the time variation of the conditional quantiles of returns which can be illustrated based on the location-scale model $r_{t+1} = \mu + \sigma_{t+1}\eta_{t+1}$ discussed in the previous Section. The quantile at level $\tau$ for this model, conditional on information available at time $t$, $\mathcal{F}_t$, is given by $q_{t+1}(\tau|\mathcal{F}_t) = \mu + \sigma_{t+1}q_\eta(\tau)$, where $q_\eta(\tau)$ represents the $\tau$-th quantile of the error term distribution. Since we assume that the conditional mean $\mu$ and the quantiles of the error $q_\eta(\tau)$ are constants, the return quantile varies over time only through changes in the conditional standard deviation of the process. We then introduce the hypothesis that the variation in the conditional variance $\sigma^2_{t+1}$ is equal to the expected realized volatility measure in day $t+1$, denoted by $E_t(RM_{t+1})$, where by $RM_{t+1}$ we denote any of the unbiased estimators of quadratic variation that were discussed in Section (2). We can assume that the variation in the expected realized measure is a function of a set of variables observable at time $t$, $X_t$, which might include past values of the measure and past returns, among others. Based on these assumptions, the conditional variance of returns becomes $\sigma^2_{t+1} = (\delta_0 + X_t'\delta_1)^2$, where $\delta_0$ and $\delta_1$ are parameters and where we assume that $\delta_0 + X_t'\delta_1 > 0$. The quantile regression model (see Koenker and Bassett, 1978) for the return in the location-scale model can thus be written as

$$q_{t+1}(\tau|\mathcal{F}_t) = \beta_0(\tau) + X_t'\beta_1(\tau) \quad (9)$$

where $\beta_0(\tau) = \mu + \delta_0 q_\eta(\tau)$ and $\beta_1(\tau) = \delta_1 q_\eta(\tau)$. Therefore, the quantile coefficients are determined both by the relationship between $\sigma^2_{t+1}$ and the explanatory variables $X_t$ as well as by the quantiles of the error term $\eta_{t+1}$. An advantage of this model is the fact that it abstracts from making an assumption about the error distribution, as it is instead the case for the parametric models discussed in the previous Section. In addition, it is also flexible in terms of the explanatory variables (or their transformations) that can be included in the vector $X_t$.

In the empirical application, we consider different specifications of the quantile regression model which use the realized volatility measures discussed in Section (2): the realized volatility ($RV_t$), the realized kernel $RV(q)_t$ for $q = 2$, the "jump-free" measure $C_t$ in Equation (7), $C_t$ together with the jump component $J_t$ in Equation (6), and $RVN_t$ which includes the overnight return. To differentiate volatility from variance in what follows, we denote with $RM_t$ a realized measure of variance (i.e., $RM_t$ is equal to $RV_t$, $RV(2)_t$, $C_t$, $[C_t,J_t]$, and $RVN_t$) and the volatility by its square root, $rm_t = \sqrt{RM_t}$. The baseline specification that we consider is the Heterogeneous Auto-Regressive (HAR) model proposed by Corsi (2009) that consists of including the (past) realized volatility measure as well as its 5-day and 22-day moving average (that proxy for a trading week and month, respectively). The smoothed measures help to account for the slowly changing components of volatility and serve as a parsimonious way to model the long memory in the series. For the HAR specification, the vector of explanatory variables in Equation (9) is thus given by $X_t = [rm_t, \overline{rm}^w_t, \overline{rm}^m_t]$, where $rm_t$ represents the square root of a realized measure in day $t$ and $\overline{rm}^w_t$ and $\overline{rm}^m_t$ represent the square root of the weekly and monthly moving average of the realized volatility $RM_t$, that is, $\overline{rm}^w_t = \sqrt{\frac{1}{5}\sum_{j=1}^{5} RM_{t+1-j}}$ and $\overline{rm}^m_t = \sqrt{\frac{1}{22}\sum_{j=1}^{22} RM_{t+1-j}}$. Using different high-frequency measures within the same HAR-type specification allows us to evaluate if any of these measures has significantly higher power to forecast the return distribution in comparison to the other measures. Given the evidence of an asymmetric effect of return surprises on volatility, we also consider two specifications that account for this characteristic within the quantile regression model. We achieve this by augmenting the HAR specification for the square root of realized volatility, $rv_t$, with a nonlinear transformation of the return and the return standardized by the square root of realized volatility given by $e_t = r_t/rv_t$. In particular, we consider the following specifications of the return quantile model in Equation (9):

- $X_t = [e_t, |e_t|, rv_t, \overline{rv}^w_t, \overline{rv}^m_t]$
- $X_t = [|r_t|, |r_t|I(r_t < 0), rv_t, \overline{rv}^w_t, \overline{rv}^m_t]$

Both specifications allow for an asymmetric effect on the quantiles depending on the return or the standardized return being positive or negative, similarly to the assumption introduced in the GJR model.

The ultimate goal of the paper is to evaluate whether the realized volatility models provide more accurate forecasts of the return quantiles and distribution relative to a GARCH-type time series model. In order to disentangle the contribution to the forecast performance that derives from the realized volatility measures as opposed to the contribution of the semi-parametric character of the quantile regression model, we also include in the analysis two specifications that replace the realized measure with the absolute return. We use the square root of the Exponential Weighted Moving Average (EWMA) of the squared returns (with smoothing parameter set to 0.94) and the HAR specification in which we use the squared returns instead of the high-frequency variance measures. The comparison of the performance of the model when using returns and realized measures allows to evaluate the informational advantage of considering the high-frequency measures of volatility when we control for the contribution of the modeling assumption. Furthermore, we also consider a market-based measure of volatility given by the VIX index in a HAR specification of the return quantile model (see Becker *et al.*, 2007, for a recent reference on the relevance of VIX to forecast volatility relative to model-based forecasts).

So far we dealt only with the case of a forecast horizon of one day, although in many applications the horizon of interest might be longer. For horizons larger than one day, we adopt a direct forecasting approach which consists of modeling the quantiles of the cumulative $h$-period return $r^h_{t+h}$ as a function of the vector $X_t$, that is,

$$q_{t+h}(\tau|\mathcal{F}_t) = \beta_{0,h}(\tau) + X_t'\beta_{1,h}(\tau) \qquad (10)$$

where the quantile parameters have now been denoted by $\beta_{0,h}$ and $\beta_{1,h}$ to stress the fact that they depend on the forecast horizon $h$. The proposed approach differs in several aspects from the existing papers connecting the return distribution to the high frequency volatility measures. The main difference is that we do not model directly the realized volatility measure, but simply use it as an explanatory variable in the quantile regression. The implication of this assumption is that we cannot use an iterative approach to generate multi-step ahead forecasts and thus use a direct approach as discussed above.

# 4 Forecast evaluation

Diebold *et al.* (1998) and Christoffersen (1998) are two early papers that propose tests to evaluate density and interval forecasts based on the assumption that the forecasting model is correctly specified. However, empirical models are likely to be misspecified so that their relative accuracy, rather than their absolute performance, might be of more interest to a forecaster. There are several approaches to compare the accuracy of distribution and density forecasts, with the main difference among them represented by the score or loss function that is assumed in the forecast evaluation.

A score function that is often used in the forecasting literature is the *Logarithmic Score* (LS) which is defined as $LS^i_{t+h} = \ln f^i_{t+h}(r^h_{t+h}|\mathcal{F}_t)$, where $f^i_{t+h}(\cdot|\mathcal{F}_t)$ indicates the density forecast[1] of the $h$-day cumulative return of model $i$ conditional on information available at the closing of day $t$ and $r^h_{t+h}$ is the realization of the cumulative return between day $t$ and day $t+h$. $LS^i_{t+h}$ represents the (logarithm) density forecast (at time $t$) evaluated at the realization of the cumulative $h$-period

[1]For the quantile regression model, we estimate the density by kernel smoothing $f^i_{t+h}(\tau|\mathcal{F}_t) = [\tau_i - \tau_{i-1}]/[q^i_{t+h}(\tau_i|\mathcal{F}_t) - q^i_{t+h}(\tau_{i-1}|\mathcal{F}_t)]$ on a grid of values for the $\tau$s.

return. When comparing two models, say model $i$ and $j$, the difference in their log-score $\Delta LS^{ij}_{t+h}$ defined as

$$\Delta LS^{ij}_{t+h} = LS^{i}_{t+h} - LS^{j}_{t+h} \tag{11}$$

provides a measure of the (relative) forecast accuracy of the two models. A positive value of $\Delta LS^{ij}_{t+h}$ indicates that model $i$ outperforms model $j$, and viceversa in case it is negative. The sample average of the log-score difference, $\overline{\Delta LS}^{ij}_{h} = \left(\sum_t \Delta LS^{ij}_{t+h}\right)/P$ (where $P$ denotes the number of forecasts), provides a statistic which can be used to test the null hypothesis of equal forecast accuracy of model $i$ and $j$ as proposed in Amisano and Giacomini (2007) and discussed later in the Section. Maheu and McCurdy (2011) and Shephard and Sheppard (2010) are some recent papers that use LS to compare the forecasting performance for the return distribution of a benchmark GARCH specification to models that incorporate realized volatility measures. However, the logarithmic score has been criticized for two reasons. First, it focuses the evaluation on the value of the densities at the realization of the variable, and does not take into account their local shape in a neighborhood of the realization. In addition, the comparison of density forecasts using LS provides an overall assessment of the performance but does not allow the evaluation to be focused on a specific area of the distribution that might be of interest to the forecaster, for instance the tails or the center of the distribution (see Gneiting and Ranjan, 2011). Moreover, forecasters might be interested in comparing models based on some characteristics of the probability distributions, such as a specific quantile. An example relevant to finance is Value-at-Risk (VaR), which represents the quantile of a portfolio or asset return distribution at the typical level of 1%. In this case, the evaluation is usually performed using the likelihood tests of Christoffersen (1998) that are based on the properties of the quantile violation process under the assumption of correct model specification, such as its unconditional/conditional coverage and its serial independence. This evaluation criteria have been considered in Kuester *et al.* (2006) and Brownlees and Gallo (2010). Instead, Clements *et al.* (2008) use a tick loss function to evaluate and compare VaR forecasts. We adopt the same approach and describe the method below.

We follow Gneiting and Raftery (2007) that propose to evaluate quantile forecasts based on the *Quantile Score* (QS); given the time $t$ quantile forecasts of model $i$, denoted by $q^{i}_{t+h}(\tau|\mathcal{F}_t)$, and the $h$-period cumulative return $r^{h}_{t+h}$, the QS is given by

$$QS^{i}_{t+h}(\tau) = \left[q^{i}_{t+h}(\tau|\mathcal{F}_t) - r^{h}_{t+h}\right]\left[\mathcal{I}(r^{h}_{t+h} \leq q^{i}_{t+h}(\tau|\mathcal{F}_t)) - \tau\right] \tag{12}$$

where $\mathcal{I}(\cdot)$ denotes the indicator function which takes value equal to 1 if its argument is true and zero otherwise. The Quantile Score function uses the asymmetric absolute loss (or tick) function adopted in quantile regression estimation to the context of out-of-sample evaluation (see also Giacomini and Komunjer, 2005). Similarly to LS, we define the score in positive orientation so that comparing two models, e.g. $i$ and $j$, we prefer the one with highest QS. Given the difference in the QS of the two models

$$\Delta QS^{ij}_{t+h}(\tau) = QS^{i}_{t+h}(\tau) - QS^{j}_{t+h}(\tau) \tag{13}$$

we conclude that, at time $t+h$, model $i$ outperforms $j$ if $\Delta QS^{ij}_{t+h}$ is positive, and viceversa if negative. Gneiting and Ranjan (2011) conduct a statistical test of equal predictive accuracy based on this loss function using the approach proposed by Amisano and Giacomini (2007). The quantile score allows to evaluate the forecasting accuracy of the models with respect to the objective of modeling a specific quantile. For example, Clements *et al.* (2008) consider the 2.5% and 5% quantiles given their interest to evaluate the left tail of the return distribution. In this paper we consider the test for $\tau$ that ranges between 0.01 and 0.99 so that we can evaluate whether some models are better at modeling a specific quantile of the distribution as opposed to other parts. While in some applications (e.g., VaR) the interest is in evaluating a single quantile, there might be other applications in which the interest is in the performance of the models in a certain area of the distribution, e.g. the left or right tail of the distribution. In this case, Gneiting and Ranjan (2011) propose a statistic that consists of integrating the score in Equation (13) across $\tau$, but with the score weighted by a function that focuses the evaluation on the area of interest. The *Weighted Quantile Score* (WQS) statistic of model $i$, denoted by $WQS^i_{t+h}$, is thus given by

$$WQS^i_{t+h} = \int_0^1 QS^i_{t+h}(\tau)\omega(\tau)d\tau \tag{14}$$

where $\omega(\tau)$ indicates a weight function in the unit interval. We consider five weight functions: (1) $\omega(\tau) = 1$ which is a uniform weight and provides an overall evaluation of the forecast distribution (an alternative to the $LS$ statistic), (2) $\omega(\tau) = \tau(1-\tau)$ gives higher weight to the central quantiles and smaller weight to the tails, (3) $\omega(\tau) = (2\tau - 1)^2$ if the focus is on the tails of the distribution, (4) $\omega(\tau) = (1-\tau)^2$ gives higher weight to the left tail of the distribution, and (5) $\omega(\tau) = \tau^2$ that focuses on the right tail. Similarly to the QS, when comparing two models using the WQS the model with higher score is preferred.

A forecaster might also be interested in producing interval forecasts and comparing the accuracy of competing models in terms of this objective. Gneiting and Raftery (2007) propose a score function suited to evaluate forecasting intervals. Define the lower and upper bound of the (central) $100 * \kappa\%$ interval by the predictive quantiles at levels $\kappa_l = (1-\kappa)/2$ as $q^i_{t+h}(\kappa_l|\mathcal{F}_t)$ and $\kappa_u = (1+\kappa)/2$ as $q^i_{t+h}(\kappa_u|\mathcal{F}_t)$. Then, the *Interval Score* (IS) is defined as

$$\begin{aligned} IS^i_{t+h}(\kappa) = \left[q^i_{t+h}(\kappa_l|\mathcal{F}_t) - q^i_{t+h}(\kappa_u|\mathcal{F}_t)\right] + \frac{2}{1-\kappa}\left(r^h_{t+h} - q^i_{t+h}(\kappa_l|\mathcal{F}_t)\right)\mathcal{I}\left(r^h_{t+h} < q^i_{t+h}(\kappa_l|\mathcal{F}_t)\right) \\ + \frac{2}{1-\kappa}\left(q^i_{t+h}(\kappa_u|\mathcal{F}_t) - r^h_{t+h}\right)\mathcal{I}\left(r_{t+h} > q^i_{t+h}(\kappa_u|\mathcal{F}_t)\right) \end{aligned} \tag{15}$$

The Interval Score penalizes wider intervals as well as observations falling outside the interval, with the penalty being a function of $\kappa$ if the observation misses the interval. Similarly to the other score functions considered, if $IS^i_{t+h}(\kappa) > IS^j_{t+h}(\kappa)$ we conclude that model $i$ is more accurate than model $j$ at time $t+h$.

To evaluate the statistical significance of the difference in performance, as measured by the score functions, we follow the approach of Giacomini and White (2006) and Amisano and Giacomini (2007). Denote by $S^i_{t+h}(\cdot)$ any of the score functions discussed above for model $i$, and by $S^j_{t+h}(\cdot)$ the score of model $j$. Then a test statistic for the null hypothesis of equal average forecast accuracy of the two models, $\overline{S}^i_{t+h}(\cdot) = \overline{S}^j_{t+h}(\cdot)$ for $t = 1, \cdots, P$, is given by

$$t = \overline{\Delta S}^{ij}_h(\cdot)/\hat{\sigma} \tag{16}$$

where $\overline{\Delta S}_h^{ij}$ denotes the difference in the sample mean of the scores and $\hat{\sigma}$ represents the HAC standard error of the difference in scores. The test statistic $t$ is asymptotically standard normal and rejections for negative values of the statistic indicate that model $j$ significantly outperforms model $i$ (and vice-versa for positive values).

# 5 Application

The intra-day dataset was provided by Price-Data and consists of five-minute prices for the S&P 500 futures (SP) and 30-year US Treasury bond futures (US) contracts from January 2, 1990 to September 9, 2009 (4958 daily observations). The five-minute prices for the SP contracts cover the time interval from 9:35 to 16:15 (EST)[2], which corresponds to 80 non-overlapping return observations per day, while the US contracts span the interval from 8:25 to 15:00 (EST), resulting in 79 intra-day returns. Figure (1) shows the time series of the daily realized variance for the two assets and Table (1) reports the summary statistics for the daily squared returns and the realized measures discussed in Section (2). The evidence indicates that the realized volatility of the equity index returns is higher and significantly more variable relative to the volatility of the bond futures returns. In addition, the Table shows that the realized measures $RV_t$, $RV(2)_t$ and $C_t$ have lower mean and variability compared to both the squared returns and $RVN_t$, which accounts for the overnight returns. We start the out-of-sample forecasting exercise on January 3, 2000 (for a total of 2419 forecasts) and the model parameters are estimated on a rolling window[3] of 2463 days (approximately 10 years of data). We consider forecast horizons of 1-, 2- and 5-day ahead while the results for 10- and 20-day are not reported because they are qualitatively similar to the results for 5-day ahead.

This Section is organized as follows. We first present the results of the comparison of the GARCH-type specifications discussed in Section (3.1). The aim is to select the most accurate forecasting model for SP and US returns to be considered as the benchmark model. We then discuss the findings on the forecast performance of the realized volatility models in comparison to the benchmark GARCH model.

## 5.1 Benchmark GARCH models

The combination of different conditional variance specifications and assumptions on the error distribution discussed in Section (3.1) produces a large number of models. We decided to limit the scope of the comparison to the following cases:

1. *GARCH(1,1)*: GARCH(1,1) model with standard normal innovations
2. *GARCH(1,1)-t*: GARCH(1,1) model with $t_k$ innovations
3. *GARCH(1,1)-EDF*: FHS with a GARCH(1,1) conditional variance

[2]We also construct realized volatility measures in the time interval between 10:00 and 15:30 (EST) which is considered more robust to microstructure frictions occurring at the beginning and at the end of the trading day. Since the results are qualitatively similar to those for the full trading day, we decided not to report these findings in the paper.

[3]We also used an expanding estimation window and found analogous results to the rolling window.

4. *EGARCH(1,1)-EDF*: FHS with a EGARCH(1,1) conditional variance
5. *GJR*: GJR-GARCH(1,1) with standard normal innovations
6. *GJR-EDF*: FHS with a GJR-GARCH(1,1) conditional variance

The first three models share the same conditional variance specification but differ in the distributional assumption for the error term. The remaining models allow for the leverage effect which is combined with normal errors or with errors resampled from the EDF. In this case we choose the number of replications $B$ equal to 10000. In comparing these volatility models, we use the GARCH(1,1)-EDF and GJR-EDF as the benchmark models against which we evaluate the remaining five models. We evaluate the density, quantile and interval forecasts generated by these models using the predictive accuracy tests discussed in Section (4). In all cases, the test statistics are standard normal distributed and rejections of the null hypothesis of equal (average) accuracy for negative values indicate that the benchmark model (GARCH(1,1)-EDF or GJR-EDF) is (significantly) outperformed by the alternative model.

**S&P 500 Return (SP)**

Table (2) shows the $t$-statistic of the Log-Score (LS) and the Weighted Quantile Score (WQS) for $h$ equal to 1, 2, and 5. Two findings emerge from the comparison of the volatility models to the GARCH(1,1)-EDF benchmark. First, the positive LS test statistics obtained by comparing the benchmark to the GARCH(1,1) model with normal and $t$ distributed errors suggest that these models are significantly less accurate (compared to the benchmark) at all forecast intervals $h$. Similar findings are provided by the WQS with uniform weight at the 1 and 5 day horizons. In addition, the WQS statistics show that, at all horizons, the GARCH(1,1)-EDF has similar performance to the models with normal and $t$ distributed errors on the left tail of the return distribution, but it is significantly more accurate on the right tail. It is thus the case that the three distributional assumptions provide similarly accurate forecasts when modeling negative returns, but the EDF assumption provides more precise forecasts of the right tail compared to the parametric distributions. This suggests that in the forecasting period 2000-2009 the (out-of-sample) evidence does not support the use of the $t$ distribution for the error term. In addition, the nonparametric nature of the EDF allows to capture some asymmetry in the error distribution which is not accounted for by the parametric distributional assumptions. Comparing the GARCH(1,1)-EDF to the EGARCH and GJR models, it appears that the LS and WQS-uniform tests are significantly negative at the 1 and 2 day horizons, but only the GJR-EDF specification outperforms the benchmark for h=5. Furthermore, at the 1-day horizon we observe rejections for negative values of the WQS focused on the left and right tails, but mostly on the right tail for h=2 and at the 5-day horizon. Hence, including a leverage effect in the conditional variance specification is important in modeling the return distribution, but this effect seems less pronounced when the object of interest is the multi-period cumulative return. When considering the GJR-EDF model as the benchmark, the Table shows that in only one case it is outperformed by the EGARCH(1,1)-EDF using the LS test for h=1, although in all other cases the test statistics are positive and mostly significant. The WQS test focused on the left and right tail shows that the GJR-EDF outperforms the alternative specifications on the right tail (as it was the case for the GARCH(1,1)-EDF case), while it provides forecast accuracy similar to the other asymmetric specifications on the left tail.

The results for the $QS(\tau)$ test in Table (3) provide a detailed analysis of the performance of the competing GARCH models in forecasting the individual quantiles. For the 1% quantiles the results suggest that all the models considered perform similarly. However, at quantile levels between 5 and 20% the GARCH(1,1)-EDF is significantly outperformed by the asymmetric specifications for $h$=1 and by the GJR specifications at the 2-day horizon. For $h$=5 all models are equally accurate. Furthermore, we find a similar pattern when looking at the top quantiles of the return distribution, with the only exception of GJR-EDF that outperforms the GARCH(1,1)-EDF also at the longest forecast horizons. In VaR applications the interest is focused on quantile levels at 1 and 5%, in which case our results indicate that the GJR-EDF model provides more accurate forecasts, in particular at the shorter horizons.

The previous discussion holds also for the IS test at the 50% and 90% level reported in Table (4). The aim in using this score rule is to compare the performance of the GARCH models in providing accurate interval forecasts. While at the 1-day horizon the GJR-EDF outperforms the GARCH(1,1) specification and has similar performance to the EGARCH one, at the longer horizons it outperforms all competing models (significantly positive test statistics). In terms of the unconditional properties of the intervals, the interval lengths are quite similar across models and they provide a coverage close to the nominal level for $h$=1, but there is a slight over-coverage for $h$=2 and 5.

Based on this evidence, the assumption on the volatility dynamics seems a much more relevant choice compared to the error distribution. We thus adopt the GJR-EDF as the benchmark in the comparison with the realized volatility models since it proved the most robust GARCH-type specification, among the ones considered, to forecast the distribution of S&P 500 futures returns.

**T-Bond Return (US)**

The results in Tables (5) to (7) for the 30-year Treasury bond futures returns provide a quite different picture compared to the findings discussed above for the S&P 500 returns. The accuracy tests show that, overall, the GARCH(1,1)-EDF and GJR-EDF provide similar forecasting performance and beat all other GARCH models, in particular at the shortest horizons examined. The fact that the two benchmarks have similar performance indicates that the evidence of a leverage effect in the bond futures returns is weaker compared to the S&P 500 returns. Considering the WQS test that focuses on the tails of the distribution, for $h$=1 the two benchmarks outperform all other models, but mostly on the right tail at the 5-day horizon. This fact can be further investigated using the $QS(\tau)$ test in Table (6) which shows that the performance of the benchmarks at $h$=1 on the left part of the return distribution is quite similar to the alternative models, although the GARCH(1,1)-EDF is outperformed by some of the models at the 5% quantile. On the right tail of the distribution, we find that the better performance of the benchmarks is mostly due to the quantile area between 0.80 and 0.99 at all forecasting horizons considered. Furthermore, the IS test shows that at the 50 and at the 90% level the GARCH(1,1)-EDF and GJR-EDF outperform most other models, and in several cases significantly so. In addition, the empirical coverage for $h$=1 is slightly higher than 50% for all models but gets closer to the nominal level at the longest horizon.

Summarizing the results for the bond futures returns, it seems that the GARCH(1,1)-EDF is a good benchmark to compare the forecast accuracy of distribution forecast from the realized volatility models since it proves to perform similarly to the GJR-EDF (which nests the GARCH(1,1)-EDF model) and outperforms the remaining models.

## 5.2 Realized volatility models

In this Section, we address the issue of the relevance of employing realized volatility measures to forecast the return distribution in comparison to using a GARCH-type volatility model. As discussed above, the selected benchmark for the S&P 500 futures returns is the GJR-EDF model, while the absence of asymmetry in the volatility process for the 30-year US T-bond returns suggests that the GARCH(1,1)-EDF is a satisfactory choice. Similarly to the previous Section, a negative test statistic indicates that the respective realized volatility model outperforms the GARCH benchmark, and the opposite when the statistic is positive.

**S&P 500 Return (SP)**

Table (8) shows that at the 1-day horizon both the LS and WQS-uniform tests indicate that using the square root of realized volatility measure, $rv_t$, in the HAR-type and the asymmetric specifications delivers significantly more accurate forecasts compared to GJR-EDF, but not when $rv_t$ is smoothed using EWMA. Moreover, the WQS also indicates that the HAR-type specifications outperform GJR on the center and right tail of the return distribution, but not significantly on the left tail. Similar results are also obtained when the high-frequency measure used is $rv(2)_t$, $c_t$, and $c_t$ jointly with the jump component $j_t$. The comparable accuracy of the models based on $c_t$ and $c_t$ together with $j_t$ thus suggests that separating the jump component does not provide higher accuracy of the distribution forecasts. We also find that the $rvn_t$ measure that accounts for the overnight return does not outperform the benchmark in any of the tests considered. In addition, the evidence does not indicate that using squared returns or the VIX index in a HAR-type specification provides better performance (compared to the time series benchmark). In fact, when using the smoothed squared returns the test statistics are significantly positive, suggesting that the distribution forecasts are less accurate compared to the GJR-EDF model. This shows that, once we control for the semi-parametric modeling assumption, high-frequency measures of volatility contain relevant information that is useful in forecasting the next day return distribution compared to using squared returns or a GARCH-type model. However, when we forecast the 2 and 5-day cumulative returns the results are less supportive of the realized volatility models. In particular, for $h$=2 some of the specifications that use $RV_t$ still outperform the GJR-EDF forecasts, but at $h$=5 there is no significant difference between the realized volatility models and the benchmark, and in fact they provide significantly worse forecasts in some cases.

In summary, the comparison using the LS and WQS-uniform tests shows that some of the realized volatility models deliver better distribution forecasts (at least at the 1-day horizon), and the analysis of the local WQS suggests that the improvement is mostly driven by their higher accuracy (compared to the benchmark) on the right tail of the return distribution. This can be further examined in Table (9) that reports the $QS(\tau)$ test: at the 1-day horizon, the realized volatility measures outperform the benchmark for quantile levels between 0.70 and 0.95, except when using $rvn_t$. Also using $VIX_t$ provides more accurate quantile forecasts than GJR-EDF on the right tail, but this is not the case when using squared returns that, in both specifications considered, have significantly positive test statistics. On the left side of the return distribution, only models that allow for the asymmetric effect of $e_t$ and $r_t$ beat the benchmark at the 0.20 and 0.30 levels. This is an interesting result since these asymmetric specifications are able to provide more accurate forecasts at both low and high quantiles relative to a benchmark model that already accounts for the asymmetric response of volatility to surprises, although in a parametric form. The $QS(\tau)$ test also shows that the only case in which incorporating the overnight return in $RVN_t$ outperforms the benchmark is for $\tau$=0.01. At the 2-day horizon, the results show that some realized volatility models outperform the GJR model at $\tau$=0.01 as well as for high quantile levels. However, at $h$=5 we do not find any evidence that the realized volatility models are more accurate compared to the benchmark.

The comparison of the interval forecasts provided in Table (10) suggests that the models based on realized measures (with the exception of $rvn_t$) outperform the benchmark at the 1-day horizon at both the 50% and 90% level. At the 2-day horizon several realized volatility models are also significant when forecasting the 90% interval, but at the 5-day horizon the GJR-EDF significantly outperforms all quantiles models for 50% intervals and has similar accuracy when forecasting 90% intervals. However, our conclusions would have been different if we adopted, as several related papers do, a GARCH(1,1) model as the benchmark. In Table (11) we present the $QS(\tau)$ test that evaluates the performance of the realized volatility models in comparison to the GARCH(1,1)-EDF benchmark. The most relevant difference, relative to the comparison with the GJR benchmark, occurs on the left tail of the distribution. In this case, all realized volatility models outperform the benchmark when forecasting the 5% quantile and most continue to be more accurate for quantile levels up to 20% for 1 and 2-day ahead forecasts, due to the misspecification of the GARCH(1,1) model regarding the leverage effect. Furthermore, the quantile model based on the squared returns performs similarly to the GARCH model at all horizons while the model based on VIX significantly outperforms the benchmark at high quantiles. These results suggest two conclusions. First, when evaluating the relative forecast accuracy of different models the choice of the benchmark is essential and the GJR-EDF model has proved a reliable option to model daily equity returns. In addition, the quantile-based model combined with the realized volatility measures proves a valuable modeling approach due to its flexible nature and adaptability to the local (in a quantile sense) dynamics of the process.

**T-Bond Return (US)**

The results for the 30-year T-Bond returns are provided in Tables (12) to (14). In this case, the return quantile models are compared to the GARCH(1,1)-EDF model that was established above as a plausible benchmark. All the statistics used in the forecast comparison point to the conclusion that realized volatility models do not outperform the GARCH(1,1) benchmark, even at the shortest horizon of 1-day. In fact, the realized volatility models are outperformed by the benchmark in the center of the distribution (as evaluated by the WQS) at the 5-day horizon. In addition, the quantile models based on the smoothed squared returns have also similar accuracy relative to the benchmark, thus suggesting that for bond returns there is no added value in considering the high-frequency volatility measures. The only case in which the quantile models outperform the benchmark is when evaluating the 1% quantile for which several realized volatility models outperform the benchmark. However, both approaches provide similar results in terms of interval coverage and average length of the intervals.

## 6 Discussion

The results discussed above suggest that the realized volatility quantile model provides more accurate distribution, quantile and interval forecasts compared to the GJR-EDF and GARCH(1,1)-EDF when predicting the S&P 500 returns at short horizons. To explain these findings, we construct measures of volatility, skewness and kurtosis that characterize the distribution of financial returns based on the estimated quantiles of the different models. An advantage of deriving these measures from the quantiles is that they inherit the property of being conditional on the information available at the time the forecast is made. In addition, the quantile-based measures are typically regarded as robust to the effect of large returns and outliers compared to averaging powers of the returns. As a quantile-based proxy for volatility we use the Interquartile Range (IQR) which is defined as $IQR_{t,h} = q_{t,h}(0.75) - q_{t,h}(0.25)$, where $IQR_{t,h}$ denotes the IQR forecast made in day $t$ at the horizon $h$ and $q_{t+h}(\tau|\mathcal{F}_t)$ represents the conditional quantile at levels $\tau = 0.25$ and 0.75. The IQR can also be interpreted as the 50% forecast interval for the returns which was statistically evaluated in the previous Section. In addition, we consider the 95% forecast interval obtained as $q_{t,h}(0.975) - q_{t,h}(0.025)$, which relies on the extreme quantiles and might highlight relevant differences between the forecasting models in the tails of the return distribution. Furthermore, we follow Kim and White (2004) and define robust skewness and kurtosis measures based on quantiles. The $h$-day ahead Skewness forecast in day $t$, denoted by $SK_{t,h}$, is defined as

$$SK_{t,h} = \frac{q_{t,h}(0.75) + q_{t,h}(0.25) - 2q_{t,h}(0.5)}{IQR_{t,h}} \tag{17}$$

while the Kurtosis forecast, denoted by $K_{t,h}$, is obtained as

$$K_{t,h} = \frac{q_{t,h}(0.975) - q_{t,h}(0.025)}{IQR_{t,h}} - 2.91 \tag{18}$$

Figure (2) shows the time series forecasts of these four quantities for the S&P 500 futures returns with $h = 1$ from January 3, 2000 until September 9, 2009 (2419 days). We confine the comparison to the GARCH(1,1)-EDF, the GJR-EDF, and the quantile model with regressors given by $rv_t, \overline{rv}_t^w$ and $\overline{rv}_t^m$. The realized volatility $RV_t$ was found to perform similarly to the other realized volatility measures. Looking at panel (a) and (b) of the Figure, it appears that the quantile model (right plot) provides wider intervals compared to the GARCH models (left and center plot) during the 2008-2009 financial instability period. In interpreting the skewness and kurtosis of the GARCH(1,1)-EDF and GJR-EDF models we should recall that the error distribution is drawn from the EDF and can thus display unconditional skewness and kurtosis, in addition to the fact that the estimation is performed on a rolling window. For both parametric models, the skewness forecast is slightly above zero at the beginning of the forecasting period and trends toward -0.1 at the end of the sample. Instead, for the quantile model the skewness forecast oscillates around zero with a persistent negative deviation in the 2005-2009 period when it reached the lowest value of -0.2. As concerns the kurtosis, the two GARCH specifications provide qualitatively similar results with the kurtosis forecasts starting above 4 at the beginning of the sample and trending downward until the end of 2005 when it increased again. On the other hand, the kurtosis forecast from the quantile model varies significantly more in the range 2.5 to 4 suggesting that the shape of the distribution forecast is more responsive to market conditions.

To further investigate the relationship between these measures and the properties of the return distribution, in Figure (3) we report non-parametric estimates of the four measures against the forecast realization, $r_{t+1}$, using a kernel smoother with a rule-of-thumb bandwidth and trimming the 1% most extreme return observations. The aim is to compare the measures derived from the different forecasting models and relate them to the statistical evidence based on the $QS(\tau)$, $WQS$ and $IS$ tests discussed in the previous Section. Panel (a) of Figure (3) shows that the realized volatility model predicts larger interquartile ranges (compared to the two parametric models) when the return realization happens to be large and negative, but only slightly larger relative to the GJR-EDF forecasts when returns are large and positive. In both tails, the GARCH(1,1)-EDF model provides ranges that are typically smaller than the other two models. In addition, the three models provide similar IQR forecasts at the center of the return distribution. It seems thus that the evidence in Table (10) indicating the higher accuracy of the 50% interval forecasts of the quantile model (compared to GJR-EDF) is driven by the fact that it predicts larger ranges when extreme (in particular negative) returns actually occur compared to the GARCH models. However, the situation is reversed when considering the 95% interval in panel (b) since the quantile model forecasts are smaller compared to both GARCH models, and in particular at the center and right tail of the return distribution. Hence, these results indicate that, in days in which the (absolute) return is large, the quantile model forecasts a distribution which is wider at the center and narrower in the tails, compared to the GARCH models. It is thus not surprising that the quantile model predicts distributions with lower kurtosis in the tails of the return distribution as shown in panel (d) of the Figure, which can be attributed to the smaller 95% intervals and larger 50% intervals (compared to GJR-EDF) that are the numerator and denominator, respectively, of the kurtosis measure defined in Equation (18). On the other hand, when the return occurs at the center of the distribution the forecasts have comparable characteristics as demonstrated by the similar kurtosis of the quantile and the GJR-EDF models. In terms of the skewness in panel (c), it seems to have no dependence with the return distribution.

To better understand these findings, in Figure (4) we report the non-parametric estimates of the relationship between the quantiles (for $\tau$ equal to 0.025, 0.25, 0.75, and 0.975) that are used in the IQR, skewness and kurtosis definitions and the forecast realization, $r_{t+1}$. Looking at panels (a) and (b) of the Figure, it appears that the wider interquartile range for large negative returns can be mostly attributed to the difference in the 75% quantile forecasts of the models considered and, to a lesser extent, to the 25%. In addition, we also find that for large positive returns, the 25% quantile of the realized volatility model and the GJR-EDF are quite similar while the 75% quantiles moderately deviate. Similarly, the fact that the 95% interval forecast of the quantile model is smaller compared to the GARCH models seems to be almost entirely driven by the quantile at $\tau = 0.025$ rather than the 97.5% quantile (panel (c) and (d) in the Figure).

The empirical analysis of the previous Section has also indicated that the quantile model with regressors given by the absolute return and its weekly and monthly moving averages is equally accurate compared to the GJR-EDF in predicting the S&P 500 returns. We thus conduct a similar analysis to the one above, in which we intend to compare the characteristics of the distribution forecasts of the quantile model with regressors $|r_t|$, $|r_t|^w$, and $|r_t|^m$ to the GARCH benchmarks. In Figure (5) we report the kernel estimates of the relationship between the IQR, 95% interval, skewness and kurtosis and the forecast realization, $r_{t+1}$, while in Figure (6) we report the selected quantiles for $\tau$ equal to 0.025, 0.25, 0.75, and 0.975. Panel (a) of Figure (5) shows that the interquartile range forecast of the GJR-EDF and the quantile model almost coincide, but differ from the GARCH(1,1)-EDF, mostly in the tails. This result is due to the fact that the 25% and the 75% quantiles of the two models are very similar as shown in panels (a) and (b) of Figure (6). The 95% interval forecasts and the kurtosis of the quantile model are slightly smaller compared to GJR-EDF (see panel (b) and (d) of Figure 5), mostly as a result of the difference of the $\tau = 0.025$ quantile. The close relationship between the quantiles (except at the lowest level) confirms the ability of the semi-parametric model to provide forecasts of similar to the best GARCH-type model, without introducing parametric assumptions on the volatility dynamics and the error distribution. Overall, these results suggest that the quantile approach provides the modeling flexibility required to account for possible asymmetries, or more generally nonlinearities, in the conditional distribution of returns, but also that considering realized measures of volatility is essential to outperform parametric models in forecasting returns.

# 7 Conclusion

The distribution of financial returns is a relevant object in financial decision-making, since it provides a complete characterization of the future outcomes and allows to derive quantities of interest to a forecaster, such as quantiles, intervals, and volatility. In this paper we adopt a quantile regression framework which represents a natural way to make the conditional distribution of future returns depend on current information. In particular, we consider realized measures of volatility as the driving force behind the variation over time of the return quantiles. The model has several advantages, among others the fact that it does not require assumptions on the error distribution as well as its adaptability in terms of the explanatory variables and their effects on different parts of the return distribution. To assess the merit of the model, we compare it to several GARCH models in which the variation of the return distribution is restricted to occur parametrically in the second moment. In addition, shifting the focus of the forecast exercise from the volatility process to the distribution allows an in-depth comparison of forecasts from different models in terms of distribution or density but also in terms of quantities like the quantiles or intervals.

A first result that emerges from our analysis is that the realized volatility quantile model outperforms an asymmetric GARCH specification in forecasting the S&P 500 returns, mostly at the 1 and 2-day horizons and on the right tail of the return distribution. On the other hand, we also find that the quantile model (significantly) outperforms the GARCH(1,1) model on both tails of the S&P 500 return distribution. These results highlight the two main characteristics of the proposed approach. The flexibility of the quantile model allows to easily adapt to the properties of the data without having to specify a parametric model and a distribution for the error term. In addition, using the realized measures of volatility provides valuable information (in a forecasting sense) compared to the information extracted from the time series of returns using a GARCH model. To further stress the relevance of the high-frequency measures to the forecasting success, we also consider a quantile model in which the realized volatility estimators are replaced by the absolute daily returns. In this case, the quantile forecasts are (in most cases) equally accurate to the GARCH benchmark models and in some cases significantly worse, hence indicating that the semi-parametric method alone does not deliver more precise distribution and quantile forecasts. On the other hand, the semi-parametric model performs similarly to the benchmark GARCH model when considering the bond returns, thus suggesting that the realized measures do not provide additional predictive content compared to a parametric volatility filter (e.g., a GARCH(1,1) model). Another fact emerging from our analysis is that the various realized measures considered in the paper perform very similarly in terms of forecast accuracy. The refinements introduced to account for microstructure noise and the separation of the continuous path variability from the jump component do not provide better performance in an out-of-sample forecasting context compared to using the simple realized volatility measure.

Overall, our results confirm the usefulness of including realized measures of volatility, in particular for the S&P 500 at short forecast horizons. However, when forecasting the equity returns at long horizons and the bond returns (at all horizons) a GARCH-type model provides reasonably accurate forecasts. These results might be considered at odds with the earlier findings in the literature that models based on realized volatility measures dominate (in- and out-of-sample) volatility forecasts obtained from parametric time series models. These opposing findings can be attributed to the difference in the object being forecast (distribution, quantile and interval rather than volatility) as well as to the loss functions adopted in the evaluation. In addition, since Andersen and Bollerslev (1998) the realized volatility literature has typically evaluated and compared realized volatility and GARCH models on their ability to forecast realized volatility rather than squared returns. This approach is not feasible in the context of quantile or interval forecasts evaluation since there are no methods available to construct realized counterparts for these quantities. In this sense, we put back the returns in the evaluation process which can (partly) explain the difference in results compared to the earlier papers. However, it can be argued that our forecast comparison is relative to a benchmark model and thus should be less sensitive to the issue as opposed to an absolute performance criterion.

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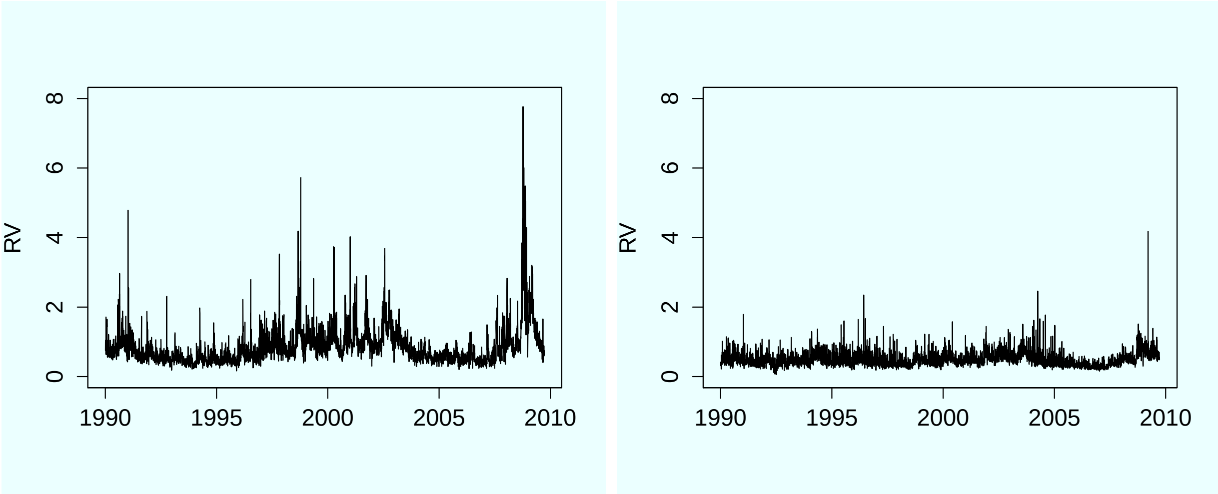



(a) S&P 500 (SP)    (b) T-Bond (US)

Figure 1: The time series of the daily realized volatility measure, $RV_t$, in Equation (1) for the S&P 500 futures returns (left) and the 30-year Treasury bond futures returns (right) from January 2, 1990 to September 9, 2009.

Table 1: **Summary Statistics**

| | $r_t^2$ | $RV_t$ | $RV(2)_t$ | $RVN_t$ | $J_t$ | $C_t$ |
|---|---|---|---|---|---|---|
| | | | SP | | | |
| Mean | 1.43 | 1.03 | 0.97 | 1.38 | 0.04 | 0.99 |
| S.D. | 4.60 | 2.13 | 2.09 | 2.74 | 0.33 | 2.04 |
| | | | US | | | |
| Mean | 0.38 | 0.29 | 0.27 | 0.39 | 0.02 | 0.26 |
| S.D. | 0.78 | 0.37 | 0.43 | 0.52 | 0.16 | 0.27 |

Mean and standard deviation (S.D.) for the daily squared returns and the realized volatility measures discussed in Section (2) for the S&P 500 futures returns (SP) and the 30-year T-bond futures returns (US).

Table 2: **S&P 500 Futures Return**

| | LS | WQS | | | | |
|---|---|---|---|---|---|---|
| | | uniform | center | tails | left | right |
| | | | | $h=1$ | | |
| *GARCH(1,1)-EDF vs*: | | | | | | |
| GARCH(1,1) | *1.76* | *2.29* | *1.87* | *2.28* | -0.13 | *3.67* |
| GARCH(1,1)-t | *2.25* | *2.59* | *2.01* | *2.84* | -0.53 | *4.59* |
| EGARCH(1,1)-EDF | **-4.43** | **-4.93** | **-4.40** | **-4.43** | **-2.70** | **-3.50** |
| GJR | **-1.79** | **-3.02** | **-2.09** | **-3.54** | **-2.30** | **-2.88** |
| GJR-EDF | **-3.01** | **-4.73** | **-4.34** | **-4.41** | **-2.27** | **-4.25** |
| *GJR-EDF vs*: | | | | | | |
| GARCH(1,1) | *3.71* | *5.87* | *5.61* | *4.96* | *1.85* | *6.11* |
| GARCH(1,1)-EDF | *3.01* | *4.73* | *4.34* | *4.41* | *2.27* | *4.25* |
| GARCH(1,1)-t | *4.00* | *5.73* | *5.37* | *5.05* | 1.62 | *6.41* |
| EGARCH(1,1)-EDF | **-1.87** | -0.66 | -1.01 | -0.32 | -0.04 | -0.41 |
| GJR | 0.70 | 1.30 | 1.13 | 1.14 | -0.66 | *2.39* |
| | | | | $h=2$ | | |
| *GARCH(1,1)-EDF vs*: | | | | | | |
| GARCH(1,1) | 1.61 | 1.05 | 0.42 | *1.87* | -0.59 | *3.01* |
| GARCH(1,1)-t | *2.01* | 1.45 | 0.64 | *2.40* | -0.73 | *4.05* |
| EGARCH(1,1)-EDF | **-2.17** | **-1.97** | **-2.74** | -1.21 | -0.19 | **-1.87** |
| GJR | **-1.72** | **-3.32** | **-3.63** | **-2.84** | **-1.86** | **-2.43** |
| GJR-EDF | **-4.06** | **-4.01** | **-3.85** | **-3.76** | -1.63 | **-4.64** |
| *GJR-EDF vs*: | | | | | | |
| GARCH(1,1) | *3.92* | *4.76* | *4.91* | *4.18* | 1.16 | *7.43* |
| GARCH(1,1)-EDF | *4.06* | *4.01* | *3.85* | *3.76* | 1.63 | *4.64* |
| GARCH(1,1)-t | *4.15* | *4.80* | *4.78* | *4.29* | 1.00 | *7.79* |
| EGARCH(1,1)-EDF | 1.37 | *1.97* | 0.68 | *2.94* | 1.58 | *2.60* |
| GJR | *1.76* | 0.47 | -0.53 | *2.33* | -1.13 | *4.00* |
| | | | | $h=5$ | | |
| *GARCH(1,1)-EDF vs*: | | | | | | |
| GARCH(1,1) | *3.12* | *1.95* | 1.18 | *3.17* | 0.40 | *4.97* |
| GARCH(1,1)-t | *3.51* | *2.26* | 1.47 | *3.53* | -0.01 | *5.75* |
| EGARCH(1,1) | 0.66 | 1.02 | 0.84 | 1.07 | 0.27 | 1.12 |
| EGARCH(1,1)-EDF | *2.83* | *3.27* | *2.85* | *3.21* | *2.50* | 1.61 |
| GJR | 0.24 | -0.95 | -1.03 | -0.76 | -0.81 | -0.24 |
| GJR-EDF | **-1.81** | **-2.15** | -1.18 | **-2.91** | -0.90 | **-3.84** |
| *GJR-EDF vs*: | | | | | | |
| GARCH(1,1) | *4.10* | *4.57* | *3.94* | *4.04* | 0.80 | *9.18* |
| GARCH(1,1)-EDF | *1.81* | *2.15* | 1.18 | 2.91 | 0.90 | *3.84* |
| GARCH(1,1)-t | *4.29* | *4.74* | *4.22* | *4.17* | 0.62 | *9.55* |
| EGARCH(1,1) | *2.21* | *2.97* | *1.67* | *4.27* | 1.23 | *4.78* |
| EGARCH(1,1)-EDF | *4.42* | *4.53* | *3.04* | *5.71* | *3.66* | *5.22* |
| GJR | *2.42* | *2.88* | 0.76 | *4.40* | 0.59 | *5.94* |

LS refers to the difference in Log Score test in Equation (11) and WQS to the difference in Weighted Quantile Score of Equation (14) for functions that weight the center, tails, left tail, and right tail of the return distribution. A negative value indicates that the alternative model outperforms the benchmark (either GARCH(1,1)-EDF or GJR-EDF), and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 3: **S&P 500 Futures Return**

| | QS($\tau$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.20 | 0.30 | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.99 |
| | | | | | | $h=1$ | | | | | | |
| *GARCH(1,1)-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | 1.08 | -0.12 | -1.26 | -1.39 | 0.16 | -0.20 | 1.22 | *2.74* | *3.18* | *2.54* | *2.08* | 1.33 |
| GARCH(1,1)-t | 1.00 | -0.58 | -1.34 | -1.47 | -0.38 | -1.06 | 1.56 | *3.01* | *3.50* | *3.64* | *3.52* | *1.75* |
| EGARCH(1,1)-EDF | 0.76 | **-3.10** | **-3.15** | **-2.90** | 0.18 | *2.44* | **-1.94** | **-3.25** | **-3.54** | **-2.88** | **-2.98** | **-2.24** |
| GJR | -0.23 | **-2.96** | **-2.57** | -1.57 | 1.03 | 1.15 | -0.86 | -1.37 | **-1.93** | **-2.20** | **-2.63** | **-1.73** |
| GJR-EDF | -0.44 | **-2.70** | **-2.39** | -1.39 | 1.23 | *1.88* | -1.59 | **-3.55** | **-4.49** | **-3.61** | **-3.26** | **-1.69** |
| *GJR-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | 1.02 | *2.27* | *1.74* | 0.15 | -0.64 | -1.31 | *3.12* | *6.85* | *7.45* | *5.39* | *4.09* | *2.25* |
| GARCH(1,1)-EDF | 0.44 | *2.70* | *2.39* | 1.39 | -1.23 | **-1.88** | 1.59 | *3.55* | *4.49* | *3.61* | *3.26* | *1.69* |
| GARCH(1,1)-t | 0.94 | *2.08* | 1.55 | 0.05 | -1.04 | **-2.00** | *3.33* | *6.89* | *7.22* | *5.74* | *4.56* | *2.28* |
| EGARCH(1,1)-EDF | 1.64 | 0.35 | -0.94 | **-1.69** | -1.51 | 0.56 | -1.12 | -0.20 | 0.56 | 0.33 | -0.72 | -0.91 |
| GJR | 0.31 | 0.12 | -1.29 | -1.14 | 0.54 | 0.44 | 0.86 | 1.50 | *2.43* | *2.01* | 1.54 | 0.23 |
| | | | | | | $h=2$ | | | | | | |
| *GARCH(1,1)-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | 0.78 | -1.03 | **-2.21** | **-2.05** | **-2.53** | -1.25 | 1.59 | 0.31 | *2.09* | *1.97* | *4.21* | *1.69* |
| GARCH(1,1)-t | 0.47 | -0.54 | **-1.76** | **-2.29** | **-3.26** | -1.51 | 1.38 | 0.78 | *2.59* | *3.04* | *5.05* | *3.18* |
| EGARCH(1,1)-EDF | *2.27* | -0.46 | -0.93 | -1.02 | -0.62 | -0.74 | -1.00 | **-3.01** | **-2.40** | **-2.04** | -1.09 | 0.46 |
| GJR | -0.68 | **-2.02** | **-1.76** | -1.45 | -0.59 | -1.02 | -1.62 | **-3.45** | **-2.74** | **-2.60** | -1.48 | -0.07 |
| GJR-EDF | -0.71 | **-1.86** | -1.63 | -1.03 | -0.64 | -0.99 | -1.18 | **-3.69** | **-3.76** | **-4.42** | **-3.46** | **-2.75** |
| *GJR-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | 0.86 | 1.36 | 1.16 | 0.15 | -0.73 | 0.16 | *3.22* | *4.08* | *5.46* | *6.66* | *5.83* | *3.76* |
| GARCH(1,1)-EDF | 0.71 | *1.86* | 1.63 | 1.03 | 0.64 | 0.99 | 1.18 | *3.69* | *3.76* | *4.42* | *3.46* | *2.75* |
| GARCH(1,1)-t | 0.63 | 1.38 | 1.07 | -0.06 | -1.18 | -0.18 | *2.81* | *4.20* | *5.56* | *7.03* | *6.28* | *4.15* |
| EGARCH(1,1)-EDF | *4.01* | 1.46 | 0.39 | -0.21 | -0.12 | 0.24 | -0.01 | -0.65 | 0.39 | *1.70* | *2.99* | *3.45* |
| GJR | -0.02 | -0.93 | -1.04 | -1.26 | -0.20 | -0.24 | -0.80 | -1.12 | 0.36 | *2.09* | *4.87* | *10.36* |
| | | | | | | $h=5$ | | | | | | |
| *GARCH(1,1)-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | 1.21 | 0.20 | -1.41 | -1.08 | **-1.73** | -0.30 | 1.12 | *1.75* | *3.15* | *5.01* | *6.19* | *4.08* |
| GARCH(1,1)-t | 1.00 | -0.10 | -1.63 | -1.47 | **-2.02** | -0.66 | *1.72* | *2.22* | *3.97* | *5.64* | *6.61* | *6.20* |
| EGARCH(1,1) | -0.83 | 0.14 | 0.90 | 0.43 | 0.22 | 0.89 | 0.15 | -0.07 | 0.28 | 0.73 | *1.68* | *1.68* |
| EGARCH(1,1)-EDF | 0.99 | *3.17* | *3.18* | *1.64* | 0.80 | 1.37 | 0.70 | 0.32 | 0.80 | 1.28 | *1.99* | *2.11* |
| GJR | -0.42 | -1.03 | -0.62 | -0.66 | -0.76 | 0.20 | -0.68 | -1.06 | -0.60 | -0.16 | 0.23 | -0.47 |
| GJR-EDF | -1.19 | -0.59 | -0.33 | -0.69 | -0.32 | -0.04 | -0.44 | -1.18 | **-1.88** | **-3.43** | **-3.97** | **-6.46** |
| *GJR-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | 1.24 | 0.52 | -0.19 | 0.28 | -0.56 | -0.30 | *2.59* | *4.08* | *5.64* | *8.66* | *8.50* | *9.01* |
| GARCH(1,1)-EDF | 1.19 | 0.59 | 0.33 | 0.69 | 0.32 | 0.04 | 0.44 | 1.18 | *1.88* | *3.43* | *3.97* | *6.46* |
| GARCH(1,1)-t | 1.15 | 0.39 | -0.34 | 0.04 | -0.81 | -0.80 | *3.41* | *4.51* | *6.29* | *8.87* | *8.63* | *9.91* |
| EGARCH(1,1) | 0.88 | 0.84 | 1.21 | 0.78 | 0.34 | 0.65 | 0.74 | 1.46 | *2.14* | *3.53* | *4.97* | *6.42* |
| EGARCH(1,1)-EDF | *3.13* | *4.18* | *3.47* | *1.79* | 0.78 | 0.88 | *1.82* | 1.89 | *2.68* | *4.21* | *5.34* | *6.88* |
| GJR | 1.13 | -0.95 | -1.28 | -0.05 | -0.40 | 0.17 | -0.22 | 0.31 | *2.26* | *4.69* | *6.09* | *6.30* |

$QS(\tau)$ refers to the difference in Quantile Score test at level $\tau$ (ranging from 0.01 to 0.99) discussed in Equation (13). A negative value indicates that the alternative model outperforms the benchmark (either GARCH(1,1)-EDF or GJR-EDF), and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 4: **S&P 500 Futures Return**

| | $\kappa = 50\%$ | | | $\kappa = 90\%$ | | |
|---|---|---|---|---|---|---|
| | IS Test | Length | Coverage | IS Test | Length | Coverage |
| | | | $h = 1$ | | | |
| *GARCH(1,1)-EDF vs* | | 1.52 | 0.52 | | 4.13 | 0.91 |
| GARCH(1,1) | 1.15 | 1.70 | 0.55 | 0.24 | 4.15 | 0.91 |
| GARCH(1,1)-t | 1.14 | 1.71 | 0.56 | 1.23 | 4.17 | 0.91 |
| EGARCH(1,1) | -0.85 | 1.83 | 0.58 | **-2.30** | 4.44 | 0.93 |
| EGARCH(1,1)-EDF | **-4.69** | 1.58 | 0.53 | **-4.02** | 4.22 | 0.91 |
| GJR | **-2.00** | 1.73 | 0.56 | **-3.36** | 4.20 | 0.91 |
| GJR-EDF | **-3.20** | 1.56 | 0.53 | **-3.36** | 4.18 | 0.91 |
| *GJR-EDF vs* | | 1.56 | 0.53 | | 4.18 | 0.91 |
| GARCH(1,1) | *4.43* | 1.70 | 0.55 | *3.26* | 4.15 | 0.91 |
| GARCH(1,1)-EDF | *3.20* | 1.52 | 0.52 | *3.36* | 4.13 | 0.91 |
| GARCH(1,1)-t | *4.04* | 1.71 | 0.56 | *3.47* | 4.17 | 0.91 |
| EGARCH(1,1) | 0.89 | 1.83 | 0.58 | 1.11 | 4.44 | 0.93 |
| EGARCH(1,1)-EDF | -1.53 | 1.58 | 0.53 | -0.64 | 4.22 | 0.91 |
| GJR | -0.24 | 1.73 | 0.56 | -0.06 | 4.20 | 0.91 |
| | | | $h = 2$ | | | |
| *GARCH(1,1)-EDF vs* | | 2.21 | 0.49 | | 5.73 | 0.92 |
| GARCH(1,1) | -0.62 | 2.38 | 0.52 | *3.00* | 5.78 | 0.92 |
| GARCH(1,1)-t | 0.15 | 2.39 | 0.52 | *3.66* | 5.82 | 0.92 |
| EGARCH(1,1)-EDF | **-2.63** | 2.40 | 0.52 | -0.93 | 6.23 | 0.94 |
| GJR | **-2.90** | 2.41 | 0.53 | **-2.27** | 5.87 | 0.93 |
| GJR-EDF | **-3.42** | 2.24 | 0.50 | **-3.03** | 5.76 | 0.92 |
| *GJR-EDF vs* | | 2.24 | 0.50 | | 5.76 | 0.92 |
| GARCH(1,1) | *2.52* | 2.38 | 0.52 | *3.50* | 5.78 | 0.92 |
| GARCH(1,1)-EDF | *3.42* | 2.21 | 0.49 | *3.03* | 5.73 | 0.92 |
| GARCH(1,1)-t | *2.78* | 2.39 | 0.52 | *3.74* | 5.82 | 0.92 |
| EGARCH(1,1)-EDF | -0.72 | 2.40 | 0.52 | *2.74* | 6.23 | 0.94 |
| GJR | -1.30 | 2.41 | 0.53 | *3.55* | 5.87 | 0.93 |
| | | | $h = 5$ | | | |
| *GARCH(1,1)-EDF vs* | | 3.51 | 0.51 | | 9.01 | 0.93 |
| GARCH(1,1) | 1.39 | 3.73 | 0.55 | *3.98* | 9.09 | 0.93 |
| GARCH(1,1)-t | 1.58 | 3.75 | 0.55 | *4.15* | 9.14 | 0.93 |
| EGARCH(1,1)-EDF | *1.88* | 4.22 | 0.59 | *4.30* | 10.96 | 0.97 |
| GJR | -1.37 | 3.79 | 0.55 | -0.47 | 9.22 | 0.94 |
| GJR-EDF | -1.33 | 3.53 | 0.52 | **-2.43** | 9.03 | 0.93 |
| *GJR-EDF vs* | | 3.53 | 0.52 | | 9.03 | 0.93 |
| GARCH(1,1) | *3.68* | 3.73 | 0.55 | *3.79* | 9.09 | 0.93 |
| GARCH(1,1)-EDF | 1.33 | 3.51 | 0.51 | *2.43* | 9.01 | 0.93 |
| GARCH(1,1)-t | *3.85* | 3.75 | 0.55 | *3.78* | 9.14 | 0.93 |
| EGARCH(1,1)-EDF | *2.68* | 4.22 | 0.59 | *5.86* | 10.96 | 0.97 |
| GJR | 0.12 | 3.79 | 0.55 | *4.25* | 9.22 | 0.94 |

IS refers to the difference in Interval Score test in Equation (4) for levels of $\kappa$ equal to 50% and 90%. *Length* indicates the average length of the interval and *Coverage* the percentage of daily returns that occur in the forecast interval. A negative value indicates that the alternative model outperforms the benchmark (either GARCH(1,1)-EDF or GJR-EDF), and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 5: **30-year Treasury Bond Futures Return**

| | LS | WQS | | | | |
|---|---|---|---|---|---|---|
| | | uniform | center | tails | left | right |
| | | | | $h=1$ | | |
| *GARCH(1,1)-EDF vs*: | | | | | | |
| GARCH(1,1) | *3.51* | *4.15* | *2.90* | *4.18* | *2.31* | *3.41* |
| GARCH(1,1)-t | *3.94* | *4.63* | *3.30* | *4.32* | 1.34 | *4.31* |
| EGARCH(1,1)-EDF | 1.00 | *2.48* | *2.47* | *2.24* | 0.29 | *3.11* |
| GJR | *3.45* | *4.10* | *2.90* | *4.12* | *2.25* | *3.38* |
| GJR-EDF | -0.62 | -0.83 | 0.12 | **-1.75** | -1.02 | **-1.94** |
| *GJR-EDF vs*: | | | | | | |
| GARCH(1,1) | *3.60* | *4.23* | *2.87* | *4.41* | *2.43* | *3.66* |
| GARCH(1,1)-EDF | 0.62 | 0.83 | -0.12 | *1.75* | 1.02 | *1.94* |
| GARCH(1,1)-t | *4.03* | *4.69* | *3.25* | *4.52* | 1.48 | *4.50* |
| EGARCH(1,1)-EDF | 1.04 | *2.50* | *2.32* | *2.37* | 0.40 | *3.25* |
| GJR | *3.57* | *4.28* | *2.89* | *4.50* | *2.44* | *3.69* |
| | | | | $h=2$ | | |
| *GARCH(1,1)-EDF vs*: | | | | | | |
| GARCH(1,1) | *3.76* | *3.85* | *1.91* | *4.59* | *3.37* | *3.23* |
| GARCH(1,1)-t | *4.14* | *4.15* | *2.46* | *4.46* | *2.78* | *3.86* |
| EGARCH(1,1)-EDF | 1.17 | *2.60* | *2.22* | *2.53* | 0.39 | *3.33* |
| GJR | *3.66* | *3.75* | *1.93* | *4.38* | *3.31* | *2.97* |
| GJR-EDF | **-1.96** | -0.44 | -0.66 | 0.10 | 0.75 | -1.00 |
| *GJR-EDF vs*: | | | | | | |
| GARCH(1,1) | *4.46* | *4.16* | *2.28* | *4.85* | *3.44* | *3.75* |
| GARCH(1,1)-EDF | *1.96* | 0.44 | 0.66 | -0.10 | -0.75 | 1.00 |
| GARCH(1,1)-t | *4.82* | *4.33* | *2.83* | *4.63* | *2.78* | *4.24* |
| EGARCH(1,1)-EDF | *1.75* | *2.65* | *2.53* | *2.39* | 0.22 | *3.51* |
| GJR | *4.40* | *4.20* | *2.32* | *4.85* | *3.53* | *3.58* |
| | | | | $h=5$ | | |
| *GARCH(1,1)-EDF vs*: | | | | | | |
| GARCH(1,1) | *2.24* | *2.21* | 1.01 | *3.42* | *2.10* | *3.17* |
| GARCH(1,1)-t | *3.34* | *1.96* | 0.94 | *3.14* | 1.42 | *3.62* |
| EGARCH(1,1)-EDF | *1.86* | *3.50* | *3.60* | *3.08* | 0.54 | *3.88* |
| GJR | *2.12* | *2.19* | 1.09 | *3.12* | *2.09* | *2.57* |
| GJR-EDF | -1.60 | 0.89 | 1.02 | 0.34 | 0.48 | -0.12 |
| *GJR-EDF vs*: | | | | | | |
| GARCH(1,1) | *2.67* | *1.81* | 0.64 | *3.46* | *2.10* | *3.11* |
| GARCH(1,1)-EDF | 1.60 | -0.89 | -1.02 | -0.34 | -0.48 | 0.12 |
| GARCH(1,1)-t | *3.56* | *1.81* | 0.70 | *3.15* | 1.36 | *3.62* |
| EGARCH(1,1)-EDF | *2.27* | *3.30* | *3.32* | *2.99* | 0.49 | *3.89* |
| GJR | *2.56* | *1.82* | 0.72 | *3.30* | *2.17* | *2.66* |

LS refers to the difference in Log Score test in Equation (11) and WQS to the difference in Weighted Quantile Score of Equation (14) for functions that weight the center, tails, left tail, and right tail of the return distribution. A negative value indicates that the alternative model outperforms the benchmark (either GARCH(1,1)-EDF or GJR-EDF), and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 6: **30-year Treasury Bond Futures Return**

| | QS($\tau$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.20 | 0.30 | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.99 |
| | | | | | | $h = 1$ | | | | | | |
| *GARCH(1,1)-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | *1.71* | **-1.69** | 0.67 | 1.55 | *2.05* | *2.08* | -0.03 | 0.14 | *3.00* | *4.52* | *3.60* | -0.55 |
| GARCH(1,1)-t | 1.59 | **-2.18** | -0.52 | 1.25 | 1.26 | *1.92* | -0.17 | 0.73 | *3.83* | *5.29* | *4.55* | -0.19 |
| EGARCH(1,1)-EDF | 0.00 | 0.25 | 0.32 | 0.21 | 1.14 | 1.18 | 0.32 | 0.42 | *1.63* | *2.97* | *2.81* | *1.99* |
| GJR | *1.69* | **-1.92** | 0.85 | 1.53 | *2.22* | *2.17* | 0.12 | -0.17 | *2.80* | *4.42* | *3.64* | -0.47 |
| GJR-EDF | -0.49 | -1.21 | -0.79 | -0.61 | 1.57 | 0.34 | 0.91 | 0.36 | -1.16 | **-2.07** | -0.60 | -1.23 |
| *GJR-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | *1.73* | -1.08 | 0.84 | 1.63 | *1.85* | *2.03* | -0.20 | 0.04 | *3.25* | *5.00* | *3.64* | -0.48 |
| GARCH(1,1)-EDF | 0.49 | 1.21 | 0.79 | 0.61 | -1.57 | -0.34 | -0.91 | -0.36 | 1.16 | *2.07* | 0.60 | 1.23 |
| GARCH(1,1)-t | 1.61 | -1.63 | -0.27 | 1.34 | 1.01 | *1.87* | -0.34 | 0.66 | *4.00* | *5.70* | *4.59* | -0.14 |
| EGARCH(1,1)-EDF | 0.03 | 0.41 | 0.44 | 0.32 | 0.75 | 1.04 | 0.10 | 0.34 | 1.80 | *3.22* | *2.79* | *2.05* |
| GJR | *1.71* | -1.56 | 1.06 | 1.62 | *2.02* | *2.13* | -0.06 | -0.27 | *3.12* | *5.05* | *3.76* | -0.40 |
| | | | | | | $h = 2$ | | | | | | |
| *GARCH(1,1)-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | *2.33* | 1.42 | 1.17 | 1.34 | *2.63* | *1.92* | -1.12 | -1.30 | 0.60 | *3.29* | *3.84* | 0.96 |
| GARCH(1,1)-t | *2.32* | 1.17 | *1.94* | 1.26 | *2.31* | 1.38 | -1.55 | 0.09 | 1.27 | *4.03* | *4.75* | 1.25 |
| EGARCH(1,1)-EDF | -0.80 | 0.45 | 1.12 | 0.41 | *2.27* | 1.55 | -1.59 | -0.16 | 1.40 | *3.97* | *3.93* | 0.91 |
| GJR | *2.30* | 1.42 | 0.94 | 1.50 | *2.82* | *2.03* | -1.25 | **-1.79** | 0.07 | *3.15* | *3.74* | 0.92 |
| GJR-EDF | 1.19 | -0.31 | 1.23 | -0.24 | -0.03 | -0.56 | -0.96 | -1.18 | 0.02 | -0.28 | -0.05 | -1.48 |
| *GJR-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | *2.29* | 1.62 | 0.06 | 1.38 | *2.55* | *2.20* | -0.52 | -0.30 | 0.64 | *3.35* | *3.83* | *3.03* |
| GARCH(1,1)-EDF | -1.19 | 0.31 | -1.23 | 0.24 | 0.03 | 0.56 | 0.96 | 1.18 | -0.02 | 0.28 | 0.05 | *1.48* |
| GARCH(1,1)-t | *2.28* | 1.34 | 0.68 | 1.33 | *2.15* | *1.82* | -0.99 | 0.65 | 1.34 | *4.08* | *4.68* | *2.58* |
| EGARCH(1,1)-EDF | -0.99 | 0.50 | 0.82 | 0.45 | *2.18* | *2.05* | -0.97 | 0.30 | 1.41 | *3.92* | *3.80* | 1.40 |
| GJR | *2.26* | *1.71* | -0.02 | 1.56 | *2.75* | *2.31* | -0.67 | -0.75 | 0.07 | *3.29* | *3.83* | *3.00* |
| | | | | | | $h = 5$ | | | | | | |
| *GARCH(1,1)-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | *1.73* | 0.70 | 0.45 | 0.89 | 0.95 | 0.71 | 0.38 | -1.32 | 0.08 | *3.46* | *3.22* | *2.95* |
| GARCH(1,1)-t | *1.72* | 0.42 | 0.49 | 1.31 | 1.17 | 0.59 | 0.00 | -0.33 | 1.25 | *4.00* | *3.84* | *4.26* |
| EGARCH(1,1)-EDF | -0.83 | 0.40 | 1.06 | *1.80* | *1.77* | *2.23* | 0.30 | 0.27 | 1.61 | *3.88* | *4.11* | *3.92* |
| GJR | *1.71* | 0.76 | 0.24 | 0.78 | 1.04 | 0.99 | 0.58 | -1.27 | -0.23 | *2.95* | *2.72* | *2.25* |
| GJR-EDF | -0.03 | 0.85 | -0.27 | 0.87 | -0.07 | 1.47 | 0.90 | -0.15 | -1.06 | 0.45 | -0.17 | 0.25 |
| *GJR-EDF vs*: | | | | | | | | | | | | |
| GARCH(1,1) | *1.83* | 0.17 | 0.70 | 0.44 | 0.96 | 0.28 | -0.04 | -1.21 | 0.67 | *3.25* | *3.06* | *2.59* |
| GARCH(1,1)-EDF | 0.03 | -0.85 | 0.27 | -0.87 | 0.07 | -1.47 | -0.90 | 0.15 | 1.06 | -0.45 | 0.17 | -0.25 |
| GARCH(1,1)-t | *1.81* | 0.11 | 0.62 | 0.69 | 1.24 | -0.04 | -0.27 | -0.30 | 1.50 | *3.93* | *3.76* | *4.05* |
| EGARCH(1,1)-EDF | -0.80 | 0.30 | 1.07 | 1.60 | *1.76* | *1.75* | 0.05 | 0.30 | *1.73* | *3.85* | *4.08* | *3.91* |
| GJR | *1.83* | 0.27 | 0.52 | 0.36 | 1.05 | 0.56 | 0.17 | -1.20 | 0.39 | *2.91* | *2.68* | *1.85* |

$QS(\tau)$ refers to the difference in Quantile Score test at level $\tau$ (ranging from 0.01 to 0.99) discussed in Equation (13). A negative value indicates that the alternative model outperforms the benchmark (either GARCH(1,1)-EDF or GJR-EDF), and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 7: **30-year Treasury Bond Futures Return**

| | $\kappa = 50\%$ | | | $\kappa = 90\%$ | | |
|---|---|---|---|---|---|---|
| | IS Test | Length | Coverage | IS Test | Length | Coverage |
| | | | $h = 1$ | | | |
| *GARCH(1,1)-EDF vs* | | 0.88 | 0.54 | | 2.23 | 0.93 |
| GARCH(1,1) | *2.36* | 0.93 | 0.56 | *2.94* | 2.27 | 0.93 |
| GARCH(1,1)-t | *2.61* | 0.93 | 0.56 | *3.64* | 2.27 | 0.93 |
| EGARCH(1,1)-EDF | 0.69 | 0.90 | 0.54 | *1.96* | 2.29 | 0.93 |
| GJR | *2.27* | 0.93 | 0.56 | *2.73* | 2.27 | 0.93 |
| GJR-EDF | 0.76 | 0.88 | 0.54 | -1.28 | 2.23 | 0.93 |
| *GJR-EDF vs* | | 0.88 | 0.54 | | 2.23 | 0.93 |
| GARCH(1,1) | *2.04* | 0.93 | 0.56 | *3.22* | 2.27 | 0.93 |
| GARCH(1,1)-EDF | -0.76 | 0.88 | 0.54 | 1.28 | 2.23 | 0.93 |
| GARCH(1,1)-t | *2.25* | 0.93 | 0.56 | *3.87* | 2.27 | 0.93 |
| EGARCH(1,1)-EDF | 0.43 | 0.90 | 0.54 | *2.04* | 2.29 | 0.93 |
| GJR | *2.01* | 0.93 | 0.56 | *3.19* | 2.27 | 0.93 |
| | | | $h = 2$ | | | |
| *GARCH(1,1)-EDF vs* | | 1.21 | 0.52 | | 3.05 | 0.91 |
| GARCH(1,1) | *1.95* | 1.26 | 0.54 | *3.46* | 3.07 | 0.91 |
| GARCH(1,1)-t | 0.77 | 1.26 | 0.54 | *4.17* | 3.08 | 0.91 |
| EGARCH(1,1)-EDF | 0.98 | 1.27 | 0.54 | *2.79* | 3.22 | 0.92 |
| GJR | *1.74* | 1.26 | 0.54 | *3.32* | 3.07 | 0.91 |
| GJR-EDF | -0.89 | 1.21 | 0.52 | -0.27 | 3.05 | 0.91 |
| *GJR-EDF vs* | | 1.21 | 0.52 | | 3.05 | 0.91 |
| GARCH(1,1) | *2.43* | 1.26 | 0.54 | *3.88* | 3.07 | 0.91 |
| GARCH(1,1)-EDF | 0.89 | 1.21 | 0.52 | 0.27 | 3.05 | 0.91 |
| GARCH(1,1)-t | 1.24 | 1.26 | 0.54 | *4.38* | 3.08 | 0.91 |
| EGARCH(1,1)-EDF | 1.28 | 1.27 | 0.54 | *2.77* | 3.22 | 0.92 |
| GJR | *2.32* | 1.26 | 0.54 | *3.84* | 3.07 | 0.91 |
| | | | $h = 5$ | | | |
| *GARCH(1,1)-EDF vs* | | 1.87 | 0.50 | | 4.70 | 0.92 |
| GARCH(1,1) | 0.22 | 1.93 | 0.52 | *2.68* | 4.72 | 0.92 |
| GARCH(1,1)-t | 0.38 | 1.94 | 0.52 | *3.01* | 4.74 | 0.92 |
| EGARCH(1,1)-EDF | *2.33* | 2.08 | 0.55 | *3.00* | 5.23 | 0.94 |
| GJR | 0.07 | 1.93 | 0.52 | *2.31* | 4.72 | 0.92 |
| GJR-EDF | 0.42 | 1.87 | 0.50 | 0.57 | 4.69 | 0.92 |
| *GJR-EDF vs* | | 1.87 | 0.50 | | 4.69 | 0.92 |
| GARCH(1,1) | 0.03 | 1.93 | 0.52 | *2.56* | 4.72 | 0.92 |
| GARCH(1,1)-EDF | -0.42 | 1.87 | 0.50 | -0.57 | 4.70 | 0.92 |
| GARCH(1,1)-t | 0.26 | 1.94 | 0.52 | *2.96* | 4.74 | 0.92 |
| EGARCH(1,1)-EDF | *2.23* | 2.08 | 0.55 | *2.86* | 5.23 | 0.94 |
| GJR | -0.12 | 1.93 | 0.52 | *2.32* | 4.72 | 0.92 |

IS refers to the difference in Interval Score test in Equation (4) for levels of $\kappa$ equal to 50% and 90%. *Length* indicates the average length of the interval and *Coverage* the percentage of daily returns that occur in the forecast interval. A negative value indicates that the alternative model outperforms the benchmark (either GARCH(1,1)-EDF or GJR-EDF), and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 8: **S&P 500 Futures Return**

| | LS | WQS | | | | |
|---|---|---|---|---|---|---|
| | | uniform | center | tails | left | right |
| *GJR-EDF vs*: | | | | $h=1$ | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.47** | **-2.18** | **-1.81** | **-2.19** | -0.69 | **-2.36** |
| $e_t$, $\lvert e_t\rvert$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.95** | **-3.38** | **-3.37** | **-2.92** | -1.31 | **-2.90** |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-3.23** | **-3.31** | **-3.47** | **-2.75** | -1.58 | **-2.76** |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | -0.77 | -0.37 | 0.16 | -0.82 | -0.76 | -0.46 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | **-2.15** | **-2.12** | -1.63 | **-2.22** | -0.93 | **-2.17** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | -1.57 | **-2.37** | **-1.98** | **-2.34** | -0.68 | **-2.55** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -0.65 | **-1.64** | -1.27 | **-1.71** | -0.22 | **-2.17** |
| $\lvert r_t\rvert^{ewma}$ | *2.98* | *4.83* | *4.96* | *4.03* | *2.46* | *4.37* |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | *4.46* | *4.31* | *2.94* | *4.81* | *2.97* | *4.83* |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -1.45 | -1.60 | -1.36 | -1.51 | 0.16 | **-2.00** |
| | | | | $h=2$ | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -1.06 | -1.29 | -0.19 | **-2.44** | -1.49 | **-1.94** |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-1.65** | **-1.99** | -0.91 | **-3.02** | -1.34 | **-3.13** |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -1.39 | **-1.94** | -1.12 | **-2.71** | -0.90 | **-3.24** |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | -0.04 | 0.53 | *1.75* | -0.84 | -1.18 | 0.05 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | -1.19 | -1.04 | 0.12 | **-2.23** | -1.47 | -1.60 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | -1.15 | -1.01 | 0.06 | **-2.19** | -1.22 | **-1.84** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 0.14 | -0.33 | 0.47 | -1.26 | -0.93 | -0.84 |
| $\lvert r_t\rvert^{ewma}$ | *3.58* | *4.51* | *5.36* | *3.60* | *2.00* | *4.90* |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | *4.13* | *4.43* | *4.72* | *3.62* | *2.34* | *3.89* |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -0.72 | -1.10 | -0.87 | -1.15 | 0.15 | **-1.73** |
| | | | | $h=5$ | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.95 | *1.82* | *2.53* | 0.70 | 0.99 | -0.51 |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *1.64* | 1.59 | *2.31* | 0.51 | 0.98 | -0.83 |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 1.53 | 1.55 | *2.21* | 0.54 | 1.05 | -0.82 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 1.55 | *2.44* | *3.26* | 1.24 | 1.19 | 0.28 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 1.13 | *1.84* | *2.54* | 0.79 | 0.98 | -0.39 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | 1.16 | *1.77* | *2.32* | 0.84 | 1.09 | -0.33 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | *1.82* | *1.88* | *2.12* | 1.27 | 1.36 | 0.26 |
| $\lvert r_t\rvert^{ewma}$ | *2.35* | *3.14* | *3.91* | *2.08* | 1.34 | *1.71* |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | *3.12* | *3.38* | *4.10* | *2.45* | *1.74* | *1.73* |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -0.73 | 0.68 | 1.05 | 0.15 | 1.15 | **-1.66** |

LS refers to the difference in Log Score test in Equation (11) and WQS to the difference in Weighted Quantile Score of Equation (14) for functions that weight the center, tails, left tail, and right tail of the return distribution. A negative value indicates that the alternative model outperforms the GJR-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 9: **S&P 500 Futures Return**

| | QS($\tau$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.20 | 0.30 | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.99 |
| *GJR-EDF vs*: | $h=1$ | | | | | | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.44 | -1.01 | -1.07 | -0.96 | -1.27 | 0.53 | -0.28 | -1.47 | **-1.98** | **-2.40** | **-2.23** | -0.93 |
| $e_t$, $\|e_t\|$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.32 | -1.19 | -1.57 | **-2.02** | **-2.74** | -1.31 | -1.21 | **-2.21** | **-2.69** | **-2.86** | **-2.56** | -1.19 |
| $\|r_t\|$, $\|r_t\|I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.06 | -1.26 | -1.39 | **-2.21** | **-2.69** | -1.53 | -1.36 | **-2.79** | **-3.03** | **-2.71** | **-2.32** | -0.82 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | **-1.80** | 0.16 | -0.27 | -0.69 | -1.14 | 0.54 | 0.69 | 0.78 | 0.34 | -0.83 | -0.88 | 0.24 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 0.17 | -1.13 | -1.48 | -0.79 | -1.61 | 0.21 | 0.06 | -0.79 | **-1.64** | **-1.92** | **-2.22** | -1.39 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | 0.03 | -0.77 | -0.85 | -0.60 | -1.15 | 0.34 | -0.13 | **-1.80** | **-2.36** | **-2.55** | **-2.19** | -1.03 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 0.48 | -0.45 | -0.74 | -0.35 | -0.35 | 1.29 | -0.17 | **-1.65** | **-1.82** | **-2.24** | **-2.08** | -0.52 |
| $\|r_t\|^{ewma}$ | 1.15 | *2.48* | *2.71* | *2.25* | 0.78 | -0.19 | *2.68* | *4.92* | *5.05* | *3.92* | *3.83* | *2.14* |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | *1.90* | *3.44* | *2.67* | 0.78 | -0.03 | -0.84 | *2.06* | *4.55* | *4.78* | *3.94* | *3.63* | *2.63* |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -0.54 | -0.03 | 0.65 | 0.52 | 0.50 | 0.79 | -1.09 | **-2.34** | **-2.40** | **-1.82** | **-1.71** | -0.76 |
| | $h=2$ | | | | | | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.16** | -1.60 | -0.93 | -0.71 | 1.38 | *2.73* | -0.21 | -1.44 | -1.44 | -1.34 | **-1.67** | **-1.68** |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.04** | -1.50 | -0.74 | -0.64 | 1.23 | *2.80* | -0.98 | **-2.50** | **-2.35** | **-2.33** | **-2.60** | **-2.34** |
| $\|r_t\|$, $\|r_t\|I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -1.10 | -1.23 | -0.50 | -0.56 | 0.84 | *2.37* | -1.35 | **-2.53** | **-2.53** | **-2.35** | **-2.68** | **-2.75** |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | **-1.72** | -1.26 | -0.69 | -0.37 | *2.01* | *2.69* | 1.53 | 0.37 | 0.74 | 0.38 | -0.36 | -0.75 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | **-2.29** | -1.50 | -0.99 | -0.59 | 1.25 | *2.49* | 0.33 | -1.03 | -1.09 | -0.98 | -1.57 | -1.36 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | **-1.68** | -1.58 | -0.91 | 0.02 | *1.92* | *2.63* | -0.52 | -1.57 | -1.39 | -1.04 | -1.50 | **-1.81** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -1.35 | -1.03 | -0.69 | -0.03 | 1.67 | *2.74* | 0.12 | -0.97 | -1.02 | -0.78 | -0.59 | -0.18 |
| $\|r_t\|^{ewma}$ | 0.95 | *2.05* | *2.16* | *1.74* | 1.50 | *2.44* | *3.00* | *4.64* | *4.18* | *4.96* | *3.78* | 1.16 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | 1.03 | *2.28* | *3.12* | *2.74* | 1.40 | *2.17* | *3.13* | *3.19* | *4.12* | *4.07* | *2.72* | 0.88 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -0.61 | -0.13 | 0.91 | 0.23 | 0.76 | *2.19* | -1.14 | **-2.68** | **-2.10** | **-1.83** | **-1.78** | 0.52 |
| | $h=5$ | | | | | | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.55 | 0.66 | 0.88 | *1.73* | *2.75* | *3.58* | *1.71* | 1.08 | 0.76 | 0.06 | -1.34 | -1.14 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.50 | 0.68 | 0.82 | *1.73* | *2.94* | *3.69* | 1.30 | 0.42 | 0.28 | -0.42 | -1.44 | -1.03 |
| $\|r_t\|$, $\|r_t\|I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.61 | 0.70 | 0.88 | *1.79* | *2.71* | *3.77* | 1.40 | 0.09 | 0.00 | -0.31 | -1.46 | -0.88 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.64 | 0.92 | 1.16 | *1.92* | *2.80* | *3.95* | *2.48* | 1.56 | 1.32 | 0.68 | -0.48 | -0.62 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 0.57 | 0.71 | 1.00 | *1.73* | *2.53* | *3.40* | *1.86* | 1.18 | 1.12 | -0.05 | -1.16 | -1.24 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | 0.65 | 0.70 | 1.01 | *1.88* | *2.67* | *3.27* | 1.56 | 0.66 | 0.46 | 0.10 | -1.11 | -0.36 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 0.74 | 1.16 | 1.18 | *2.01* | *2.59* | *2.84* | 1.31 | 0.81 | 0.81 | 0.51 | -0.60 | 0.41 |
| $\|r_t\|^{ewma}$ | 1.29 | 1.09 | 0.90 | *1.83* | *2.58* | *2.98* | *2.83* | *2.15* | *2.17* | *1.70* | 0.98 | 1.25 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | 1.31 | 1.53 | 1.63 | *2.27* | *2.84* | *3.50* | *2.87* | *1.99* | *2.12* | *1.68* | 0.88 | 1.57 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 0.91 | 0.81 | 0.98 | *1.72* | *2.30* | *2.84* | 0.53 | -1.02 | -1.56 | -1.43 | **-1.94** | -0.77 |

$QS(\tau)$ refers to the difference in Quantile Score test at level $\tau$ (ranging from 0.01 to 0.99) discussed in Equation (13). A negative value indicates that the alternative model outperforms the GJR-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 10: **S&P 500 Futures Return**

| | $\kappa = 50\%$ | | | $\kappa = 90\%$ | | |
|---|---|---|---|---|---|---|
| | IS Test | Length | Coverage | IS Test | Length | Coverage |
| | | | $h = 1$ | | | |
| *GJR-EDF vs* | | 1.56 | 0.53 | | 4.18 | 0.91 |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.91** | 1.54 | 0.51 | **-2.55** | 3.90 | 0.90 |
| $e_t$, $\|e_t\|$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-3.98** | 1.55 | 0.52 | **-2.73** | 3.89 | 0.90 |
| $\|r_t\|$, $\|r_t\|I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-3.50** | 1.55 | 0.52 | **-2.34** | 3.89 | 0.90 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | -0.32 | 1.55 | 0.52 | -0.79 | 4.00 | 0.90 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | **-2.33** | 1.56 | 0.52 | **-2.65** | 3.94 | 0.90 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | **-2.70** | 1.54 | 0.51 | **-2.55** | 3.89 | 0.89 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | **-2.20** | 1.54 | 0.51 | **-2.08** | 3.90 | 0.90 |
| $\|r_t\|^{ewma}$ | *4.04* | 1.56 | 0.53 | *3.19* | 4.13 | 0.90 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | *1.84* | 1.55 | 0.52 | *3.92* | 4.09 | 0.90 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | **-2.08** | 1.53 | 0.51 | -1.52 | 3.91 | 0.89 |
| | | | $h = 2$ | | | |
| *GJR-EDF vs* | | 2.24 | 0.50 | | 5.76 | 0.92 |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -1.11 | 2.25 | 0.49 | **-1.80** | 5.37 | 0.90 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.06** | 2.26 | 0.49 | **-2.19** | 5.34 | 0.90 |
| $\|r_t\|$, $\|r_t\|I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.14** | 2.26 | 0.49 | **-2.07** | 5.34 | 0.90 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.57 | 2.27 | 0.49 | -0.94 | 5.45 | 0.90 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | -1.12 | 2.26 | 0.49 | **-1.68** | 5.42 | 0.91 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | -0.94 | 2.23 | 0.49 | **-1.78** | 5.35 | 0.90 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -0.26 | 2.24 | 0.49 | -0.95 | 5.33 | 0.90 |
| $\|r_t\|^{ewma}$ | *3.90* | 2.27 | 0.50 | *2.84* | 5.67 | 0.91 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | *3.83* | 2.23 | 0.50 | *2.71* | 5.57 | 0.91 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | **-2.00** | 2.22 | 0.49 | -1.13 | 5.27 | 0.90 |
| | | | $h = 5$ | | | |
| *GJR-EDF vs* | | 3.53 | 0.52 | | 9.03 | 0.93 |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *2.75* | 3.30 | 0.49 | -0.08 | 7.94 | 0.89 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *2.47* | 3.31 | 0.49 | -0.13 | 7.89 | 0.89 |
| $\|r_t\|$, $\|r_t\|I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *2.11* | 3.30 | 0.49 | -0.16 | 7.88 | 0.89 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | *3.07* | 3.32 | 0.49 | 0.63 | 8.04 | 0.90 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | *2.97* | 3.32 | 0.49 | 0.12 | 8.02 | 0.90 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | *2.47* | 3.29 | 0.49 | 0.00 | 7.91 | 0.89 |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | *2.31* | 3.28 | 0.49 | 0.58 | 7.94 | 0.89 |
| $\|r_t\|^{ewma}$ | *3.18* | 3.38 | 0.51 | 1.50 | 8.31 | 0.91 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | *3.47* | 3.34 | 0.50 | *1.83* | 8.20 | 0.91 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 0.36 | 3.30 | 0.48 | -0.31 | 7.84 | 0.89 |

IS refers to the difference in Interval Score test in Equation (4) for levels of $\kappa$ equal to 50% and 90%. *Length* indicates the average length of the interval and *Coverage* the percentage of daily returns that occur in the forecast interval. A negative value indicates that the alternative model outperforms the GJR-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 11: **S&P 500 Futures Return**

| | QS($\tau$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.20 | 0.30 | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.99 |
| *GARCH(1,1)-EDF vs*: | | | | | | $h=1$ | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.23 | **-2.82** | **-2.77** | **-1.86** | -0.37 | *1.76* | -1.05 | **-2.85** | **-3.69** | **-4.07** | **-3.30** | **-1.96** |
| $e_t$, $\lvert e_t\rvert$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.08 | **-2.79** | **-3.02** | **-2.68** | **-2.02** | -0.58 | **-1.82** | **-3.53** | **-4.36** | **-4.46** | **-3.60** | **-2.35** |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.28 | **-2.63** | **-2.62** | **-2.62** | **-2.17** | -1.03 | **-1.85** | **-3.79** | **-4.50** | **-4.04** | **-3.31** | **-1.97** |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | **-2.58** | **-2.02** | **-2.62** | **-1.72** | -0.23 | 1.60 | -0.29 | -1.54 | **-2.49** | **-3.39** | **-2.73** | -1.43 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | -0.04 | **-2.92** | **-3.27** | **-1.81** | -0.86 | 1.38 | -0.78 | **-2.35** | **-3.30** | **-3.70** | **-3.31** | **-2.46** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | -0.22 | **-2.64** | **-2.60** | -1.63 | -0.26 | 1.59 | -0.97 | **-3.23** | **-4.04** | **-4.20** | **-3.29** | **-2.15** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 0.27 | **-2.20** | **-2.42** | -1.27 | 0.56 | *2.43* | -0.93 | **-3.02** | **-3.56** | **-3.92** | **-3.11** | **-1.64** |
| $\lvert r_t\rvert^{ewma}$ | 1.14 | 1.40 | *2.12* | *2.33* | *2.67* | 0.91 | 2.14 | 1.59 | 1.98 | 1.35 | 2.60 | 1.41 |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | *2.25* | *2.65* | *1.78* | 0.10 | 0.51 | -0.24 | 0.90 | 1.23 | 1.12 | 1.35 | 0.76 | 1.56 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -0.73 | **-1.98** | -0.69 | -0.33 | 1.29 | 1.57 | **-1.67** | **-3.44** | **-4.14** | **-3.54** | **-2.88** | **-1.91** |
| | | | | | | $h=2$ | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.78** | **-4.16** | **-2.67** | -1.57 | 1.04 | *1.94* | -0.71 | **-2.90** | **-2.90** | **-3.77** | **-4.04** | **-4.01** |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-2.50** | **-3.53** | **-2.22** | -1.44 | 0.91 | *2.02* | -1.35 | **-3.81** | **-3.67** | **-4.43** | **-4.80** | **-4.87** |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -1.59 | **-3.02** | **-1.84** | -1.33 | 0.55 | *1.82* | **-1.64** | **-3.60** | **-3.67** | **-4.22** | **-4.58** | **-5.14** |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | **-2.35** | **-4.28** | **-2.75** | -1.23 | *1.75* | *1.86* | 0.60 | **-1.91** | **-1.75** | **-2.73** | **-3.01** | **-2.79** |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | **-2.72** | **-4.04** | **-2.65** | -1.52 | 0.92 | *1.72* | -0.26 | **-2.61** | **-2.76** | **-3.64** | **-4.05** | **-4.17** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | **-2.27** | **-4.02** | **-2.56** | -0.88 | *1.65* | *1.87* | -0.96 | **-3.06** | **-2.98** | **-3.55** | **-4.06** | **-4.76** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | **-1.88** | **-3.24** | *-2.19* | -0.82 | 1.32 | *1.96* | -0.38 | **-2.37** | **-2.52** | **-3.15** | **-2.85** | **-1.76** |
| $\lvert r_t\rvert^{ewma}$ | 0.86 | *1.81* | *2.12* | *1.69* | *1.71* | *1.67* | *2.11* | 1.25 | *2.31* | *2.28* | *2.31* | -1.31 |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | 1.04 | *2.13* | *3.39* | *2.78* | 1.32 | *1.69* | *1.98* | -0.01 | 1.28 | 0.02 | -0.73 | -1.11 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | -0.99 | **-1.64** | -0.26 | -0.49 | 0.49 | *1.67* | -1.42 | **-3.69** | **-3.20** | **-3.72** | **-3.47** | -0.34 |
| | | | | | | $h=5$ | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.46 | 0.48 | 0.79 | 1.27 | *2.02* | *2.49* | 1.08 | 0.09 | -0.74 | **-2.61** | **-4.72** | **-3.36** |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.62 | 0.50 | 0.72 | 1.27 | *2.19* | *2.60* | 0.75 | -0.50 | -1.23 | **-2.98** | **-4.70** | **-3.10** |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.45 | 0.53 | 0.78 | 1.36 | *2.04* | *2.69* | 0.84 | -0.75 | -1.41 | **-2.61** | **-4.32** | **-2.89** |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | -0.31 | 0.87 | 1.15 | 1.51 | *2.05* | *2.72* | 1.67 | 0.43 | -0.18 | **-2.09** | **-3.61** | **-3.04** |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | -0.08 | 0.57 | 0.94 | 1.32 | *1.91* | *2.45* | 1.25 | 0.24 | -0.34 | **-2.92** | **-4.68** | **-3.82** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$ | -0.16 | 0.53 | 0.93 | 1.43 | *1.97* | *2.34* | 0.99 | -0.24 | -0.99 | **-2.38** | **-3.96** | **-1.99** |
| $c_t$, $\overline{c}^w$, $\overline{c}^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -0.36 | 1.14 | 1.11 | 1.53 | *1.89* | *2.11* | 0.91 | 0.06 | -0.31 | -1.40 | **-2.45** | -0.57 |
| $\lvert r_t\rvert^{ewma}$ | 1.23 | 1.19 | 0.96 | 1.63 | *2.19* | *2.34* | *2.69* | *2.39* | *1.77* | -1.17 | **-2.05** | -0.66 |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | 1.29 | *1.70* | *1.77* | *2.02* | *2.46* | *2.67* | *2.27* | 1.46 | 1.01 | -0.70 | -1.58 | 0.21 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 0.34 | 0.76 | 0.98 | 1.46 | *1.90* | *2.20* | 0.23 | -1.59 | **-2.44** | **-3.42** | **-4.44** | **-2.41** |

$QS(\tau)$ refers to the difference in Quantile Score test at level $\tau$ (ranging from 0.01 to 0.99) discussed in Equation (13). A negative value indicates that the alternative model outperforms the GARCH(1,1)-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 12: **30-year Treasury Bond Futures Return**

| | LS | WQS | | | | |
|---|---|---|---|---|---|---|
| | | uniform | center | tails | left | right |
| *GARCH(1,1)-EDF vs*: | | | | $h = 1$ | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.76 | 0.42 | 0.91 | -0.09 | -0.83 | 0.72 |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -1.14 | 0.16 | 0.36 | -0.06 | -0.72 | 0.62 |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.94 | 0.01 | 0.09 | -0.08 | -0.72 | 0.57 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | -0.53 | 0.31 | 1.07 | -0.65 | 0.02 | -1.22 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | -0.61 | 0.76 | 1.15 | 0.30 | -0.79 | 1.09 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$ | -1.10 | -0.08 | 0.76 | -0.84 | -0.92 | -0.12 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -1.38 | -0.59 | 0.44 | -1.57 | -0.84 | -1.35 |
| $\lvert r_t\rvert^{ewma}$ | 0.29 | 1.55 | *2.19* | 0.34 | -0.74 | 1.33 |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | 0.66 | 0.89 | 0.39 | 1.30 | 0.73 | 0.96 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | *2.69* | *2.46* | *1.83* | *2.64* | *2.58* | 1.08 |
| | | | | $h = 2$ | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.42 | *1.68* | 1.80 | 1.15 | 0.86 | 0.57 |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.68 | 0.94 | 0.94 | 0.69 | 0.39 | 0.47 |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.62 | 0.49 | 0.49 | 0.37 | 0.09 | 0.36 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.47 | 0.56 | 0.81 | -0.11 | 1.18 | **-1.73** |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 0.91 | *1.95* | *2.05* | 1.45 | 0.87 | 1.04 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$ | -0.45 | 0.94 | 1.22 | 0.34 | 0.38 | 0.01 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -0.16 | 0.04 | 0.38 | -0.45 | 0.24 | -0.94 |
| $\lvert r_t\rvert^{ewma}$ | *2.55* | *2.11* | *1.68* | *2.06* | 1.52 | 1.51 |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | *3.13* | *1.74* | 1.00 | *2.40* | *1.70* | 1.56 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | *2.09* | 1.46 | 0.48 | *2.61* | *2.59* | 0.72 |
| | | | | $h = 5$ | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 1.03 | *2.12* | *2.46* | 1.19 | 0.90 | 0.65 |
| $e_t$, $\lvert e_t\rvert$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.99 | *1.92* | *2.19* | 1.16 | 0.88 | 0.67 |
| $\lvert r_t\rvert$, $\lvert r_t\rvert I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 1.02 | *1.92* | *2.25* | 1.15 | 0.89 | 0.67 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.94 | *1.68* | *1.64* | 1.02 | 1.14 | 0.09 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 0.83 | *1.99* | *2.34* | 1.13 | 0.58 | 0.89 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$ | 0.92 | *1.72* | *2.12* | 0.78 | 0.59 | 0.43 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 1.36 | 1.04 | 1.04 | 0.67 | 0.77 | 0.09 |
| $\lvert r_t\rvert^{ewma}$ | 0.16 | *1.77* | *1.71* | 1.24 | 1.02 | 0.85 |
| $\lvert r_t\rvert$, $\overline{\lvert r_t\rvert}^w$, $\overline{\lvert r_t\rvert}^m$ | 1.48 | *1.78* | 1.43 | *1.74* | *1.60* | 0.84 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 1.18 | 1.10 | 0.67 | 1.59 | 1.42 | 0.58 |

LS refers to the difference in Log Score test in Equation (11) and WQS to the difference in Weighted Quantile Score of Equation (14) for functions that weight the center, tails, left tail, and right tail of the return distribution. A negative value indicates that the alternative model outperforms the GARCH(1,1)-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

Table 13: **30-year Treasury Bond Futures Return**

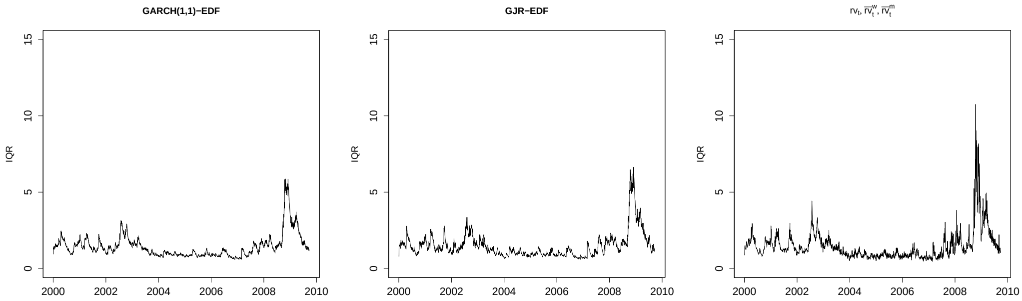
| | QS($\tau$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.20 | 0.30 | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.99 |
| *GARCH(1,1)-EDF vs* | $h=1$ | | | | | | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-1.78** | -1.20 | 0.15 | 0.36 | 1.30 | *1.83* | 0.05 | 0.11 | -0.03 | 0.18 | 0.21 | *1.67* |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-1.74** | -0.76 | 0.36 | 0.27 | 0.69 | 0.84 | -0.52 | -0.13 | -0.32 | 0.00 | 0.39 | 1.53 |
| $\|r_t\|$, $\|r_t\|I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | **-1.83** | -0.78 | 0.36 | 0.43 | 0.69 | 0.60 | -0.86 | -0.35 | -0.37 | -0.19 | 0.32 | *1.73* |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | -0.95 | 0.08 | 0.27 | 0.76 | *1.86* | *2.47* | 0.31 | 0.29 | -0.60 | -0.97 | -1.26 | -0.93 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | **-1.92** | -0.81 | 0.17 | 0.15 | 0.62 | *2.03* | 0.14 | 0.69 | 0.42 | 0.26 | 0.62 | *1.86* |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$ | **-1.72** | -1.27 | -0.11 | 0.10 | *1.66* | *2.03* | 0.15 | -0.33 | -0.63 | -0.75 | -0.65 | 1.38 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | -1.33 | -1.48 | -0.22 | 0.48 | *1.67* | *1.94* | 0.19 | -0.85 | **-1.70** | **-2.03** | -1.43 | 0.96 |
| $\|r_t\|^{ewma}$ | **-1.76** | -0.38 | -0.34 | -0.05 | *1.73* | *2.31* | 0.82 | *3.41* | *2.04* | 0.83 | 0.45 | 0.90 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | -1.05 | 1.30 | 0.96 | 0.89 | 0.45 | 0.51 | -0.41 | 0.12 | -0.68 | 0.36 | 0.83 | *1.76* |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | *1.90* | *1.76* | *2.58* | *2.15* | *1.79* | 0.55 | 0.52 | 1.34 | 0.29 | 0.56 | 0.48 | *1.97* |
| | $h=2$ | | | | | | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.96 | 0.07 | 0.97 | 1.16 | 0.40 | 0.98 | 1.11 | 0.60 | 1.19 | 0.54 | -0.10 | 0.66 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.99 | -0.27 | 0.00 | 0.72 | 0.28 | 0.60 | 0.50 | -0.15 | 0.54 | 0.57 | 0.00 | 0.61 |
| $\|r_t\|$, $\|r_t\|I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.61 | -0.61 | 0.27 | 0.30 | -0.22 | 0.30 | 0.24 | -0.24 | 0.39 | 0.36 | -0.10 | 0.72 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 1.03 | 0.74 | 0.83 | 1.50 | 0.75 | 0.79 | 0.57 | 0.00 | -0.20 | **-1.99** | **-2.27** | -0.63 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 1.07 | 0.09 | 0.80 | 1.13 | 0.58 | 0.97 | *1.89* | 1.16 | *1.67* | 1.12 | 0.23 | 0.84 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$ | 0.25 | 0.02 | 0.50 | 0.71 | 0.29 | 0.88 | 0.58 | 0.20 | 0.38 | -0.07 | -0.55 | 0.50 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 0.56 | -0.24 | 0.08 | 0.48 | 0.54 | 0.72 | 0.19 | -1.15 | -0.58 | -1.07 | -1.10 | 0.08 |
| $\|r_t\|^{ewma}$ | *1.72* | 0.62 | 0.94 | 1.08 | 0.67 | 0.59 | 1.42 | *2.10* | *2.17* | 1.31 | 0.67 | 1.23 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | 1.26 | 1.16 | *1.80* | 0.92 | 0.34 | -0.06 | 0.13 | 1.05 | *1.68* | 1.04 | 0.92 | *1.75* |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | *1.71* | *2.51* | *2.75* | 1.50 | 0.22 | -0.48 | -0.53 | -0.32 | 0.12 | 0.65 | 0.11 | 1.49 |
| | $h=5$ | | | | | | | | | | | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.41 | 0.74 | 0.68 | 1.51 | 1.12 | 1.22 | 1.59 | *2.14* | *1.71* | 0.91 | -0.23 | 0.43 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.38 | 0.68 | 0.72 | 1.54 | 1.18 | 0.84 | 1.20 | *1.93* | *1.64* | 0.87 | -0.16 | 0.56 |
| $\|r_t\|$, $\|r_t\|I(r_t<0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.61 | 0.52 | 0.58 | 1.59 | 1.33 | 0.83 | 1.30 | *2.01* | 1.59 | 0.84 | -0.14 | 0.57 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.24 | 1.07 | 0.98 | *1.86* | 1.45 | 1.50 | 0.90 | 1.13 | 0.79 | 0.38 | -0.61 | -0.02 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 0.11 | 0.43 | 0.48 | 1.14 | 0.88 | 1.15 | 1.51 | *2.26* | *1.95* | 1.15 | 0.00 | 0.50 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$ | 0.25 | 0.25 | 0.35 | 1.48 | 1.30 | 1.41 | 1.26 | *1.93* | 1.57 | 0.62 | -0.55 | 0.44 |
| $c_t$, $\overline{c}_t^w$, $\overline{c}_t^m$, $j_t$, $\overline{j}_t^w$, $\overline{j}_t^m$ | 0.69 | 0.40 | 0.44 | 1.25 | 1.07 | 0.91 | -0.06 | 0.64 | 1.06 | 0.13 | -0.81 | 0.44 |
| $\|r_t\|^{ewma}$ | -0.03 | 1.02 | 0.79 | 1.47 | 1.07 | 1.30 | 1.14 | *1.76* | *2.28* | 1.00 | 0.08 | 0.06 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | 1.21 | 1.48 | 1.34 | *1.90* | 1.21 | 0.79 | 0.61 | 1.10 | *1.97* | 0.89 | 0.16 | 0.67 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 0.19 | *1.78* | 1.21 | 1.19 | 1.14 | 0.63 | -0.47 | -0.03 | 1.06 | 0.77 | -0.17 | 0.71 |

$QS(\tau)$ refers to the difference in Quantile Score test at level $\tau$ (ranging from 0.01 to 0.99) discussed in Equation (13). A negative value indicates that the alternative model outperforms the GARCH(1,1)-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.

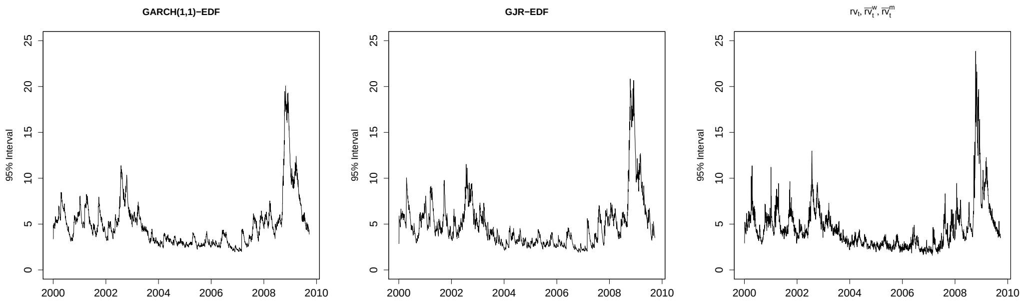
Table 14: **30-year Treasury Bond Futures Return**

| | $\kappa = 50\%$ | | | $\kappa = 90\%$ | | |
|---|---|---|---|---|---|---|
| | IS Test | Length | Coverage | IS Test | Length | Coverage |
| | | | $h = 1$ | | | |
| *GARCH(1,1)-EDF vs*: | | 0.88 | 0.54 | | 2.23 | 0.93 |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.53 | 0.85 | 0.52 | -0.67 | 2.14 | 0.91 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.10 | 0.85 | 0.52 | -0.21 | 2.14 | 0.92 |
| $\|r_t\|$, $\|r_t\|I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.10 | 0.85 | 0.52 | -0.26 | 2.14 | 0.92 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.79 | 0.89 | 0.53 | -0.68 | 2.21 | 0.93 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 0.47 | 0.86 | 0.52 | -0.01 | 2.15 | 0.91 |
| $c_t$, $\bar{c}_t^w$, $\bar{c}_t^m$ | 0.08 | 0.86 | 0.52 | -1.52 | 2.15 | 0.92 |
| $c_t$, $\bar{c}_t^w$, $\bar{c}_t^m$, $j_t$, $\bar{j}_t^w$, $\bar{j}_t^m$ | -0.16 | 0.86 | 0.53 | **-2.23** | 2.14 | 0.92 |
| $\|r_t\|^{ewma}$ | *1.78* | 0.89 | 0.54 | 0.13 | 2.21 | 0.93 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | -0.25 | 0.88 | 0.53 | 1.50 | 2.19 | 0.92 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 1.27 | 0.85 | 0.52 | 1.29 | 2.14 | 0.91 |
| | | | $h = 2$ | | | |
| *GARCH(1,1)-EDF vs*: | | 1.21 | 0.52 | | 3.05 | 0.91 |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.96 | 1.15 | 0.51 | -0.03 | 2.92 | 0.90 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | 0.11 | 1.15 | 0.50 | -0.18 | 2.92 | 0.90 |
| $\|r_t\|$, $\|r_t\|I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | -0.33 | 1.15 | 0.50 | -0.44 | 2.93 | 0.90 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | 0.57 | 1.20 | 0.52 | -0.84 | 3.01 | 0.91 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | 1.25 | 1.16 | 0.50 | 0.24 | 2.93 | 0.90 |
| $c_t$, $\bar{c}_t^w$, $\bar{c}_t^m$ | 0.16 | 1.16 | 0.51 | -0.45 | 2.93 | 0.90 |
| $c_t$, $\bar{c}_t^w$, $\bar{c}_t^m$, $j_t$, $\bar{j}_t^w$, $\bar{j}_t^m$ | -0.56 | 1.17 | 0.51 | -1.08 | 2.92 | 0.90 |
| $\|r_t\|^{ewma}$ | *1.72* | 1.20 | 0.52 | 0.89 | 3.01 | 0.91 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | 1.07 | 1.19 | 0.52 | 1.38 | 2.99 | 0.91 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 0.21 | 1.14 | 0.50 | *1.86* | 2.94 | 0.90 |
| | | | $h = 5$ | | | |
| *GARCH(1,1)-EDF vs*: | | 1.87 | 0.50 | 4.70 | 0.92 | |
| $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *2.78* | 1.78 | 0.48 | 0.24 | 4.33 | 0.89 |
| $e_t$, $\|e_t\|$,$rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *2.96* | 1.77 | 0.48 | 0.23 | 4.32 | 0.89 |
| $\|r_t\|$, $\|r_t\|I(r_t < 0)$, $rv_t$, $\overline{rv}_t^w$, $\overline{rv}_t^m$ | *3.03* | 1.78 | 0.48 | 0.16 | 4.32 | 0.89 |
| $rvn_t$, $\overline{rvn}_t^w$, $\overline{rvn}_t^m$ | *2.06* | 1.84 | 0.50 | 0.30 | 4.46 | 0.90 |
| $rv(2)_t$, $\overline{rv(2)}_t^w$, $\overline{rv(2)}_t^m$ | *2.76* | 1.78 | 0.49 | 0.22 | 4.35 | 0.89 |
| $c_t$, $\bar{c}_t^w$, $\bar{c}_t^m$ | *2.75* | 1.78 | 0.49 | -0.22 | 4.35 | 0.89 |
| $c_t$, $\bar{c}_t^w$, $\bar{c}_t^m$, $j_t$, $\bar{j}_t^w$, $\bar{j}_t^m$ | *2.12* | 1.78 | 0.49 | -0.26 | 4.35 | 0.89 |
| $\|r_t\|^{ewma}$ | *2.59* | 1.83 | 0.50 | 0.47 | 4.48 | 0.90 |
| $\|r_t\|$, $\overline{\|r_t\|}^w$, $\overline{\|r_t\|}^m$ | *2.53* | 1.82 | 0.50 | 0.93 | 4.43 | 0.90 |
| $VIX_t$, $\overline{VIX}_t^w$, $\overline{VIX}_t^m$ | 1.46 | 1.79 | 0.49 | 1.14 | 4.38 | 0.89 |

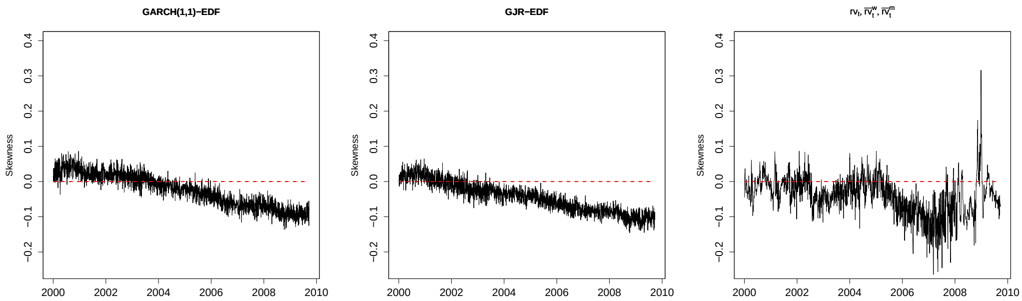
IS refers to the difference in Interval Score test in Equation (4) for levels of $\kappa$ equal to 50% and 90%. *Length* indicates the average length of the interval and *Coverage* the percentage of daily returns that occur in the forecast interval. A negative value indicates that the alternative model outperforms the GARCH(1,1)-EDF benchmark, and the opposite for a positive value. The value reported is the t-statistic as described in Equation (16) which is standard normal distributed. In bold are denoted the test statistics that are significantly negative at 5% (one-sided), and in italics the ones that are significantly positive at the same level. $h$ denotes the forecasting horizon.



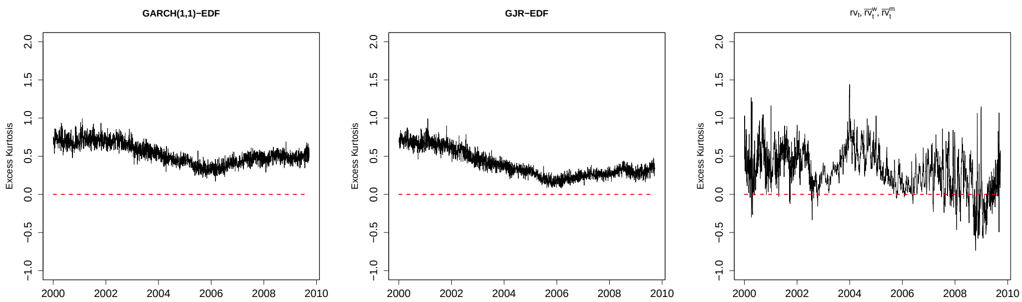

(a) Interquartile Range




(b) 95% Interval




(c) Skewness




(d) Kurtosis

Figure 2: Time series of the estimated $IQR_{t,1}$, 95% interval forecast, $SK_{t,1}$ and $K_{t,1}$ for the GARCH(1,1)-EDF (left column), GJR-EDF (center), and quantile model with regressors $X_t = [rv_t, \overline{rv}_t^w, \overline{rv}_t^m]$.

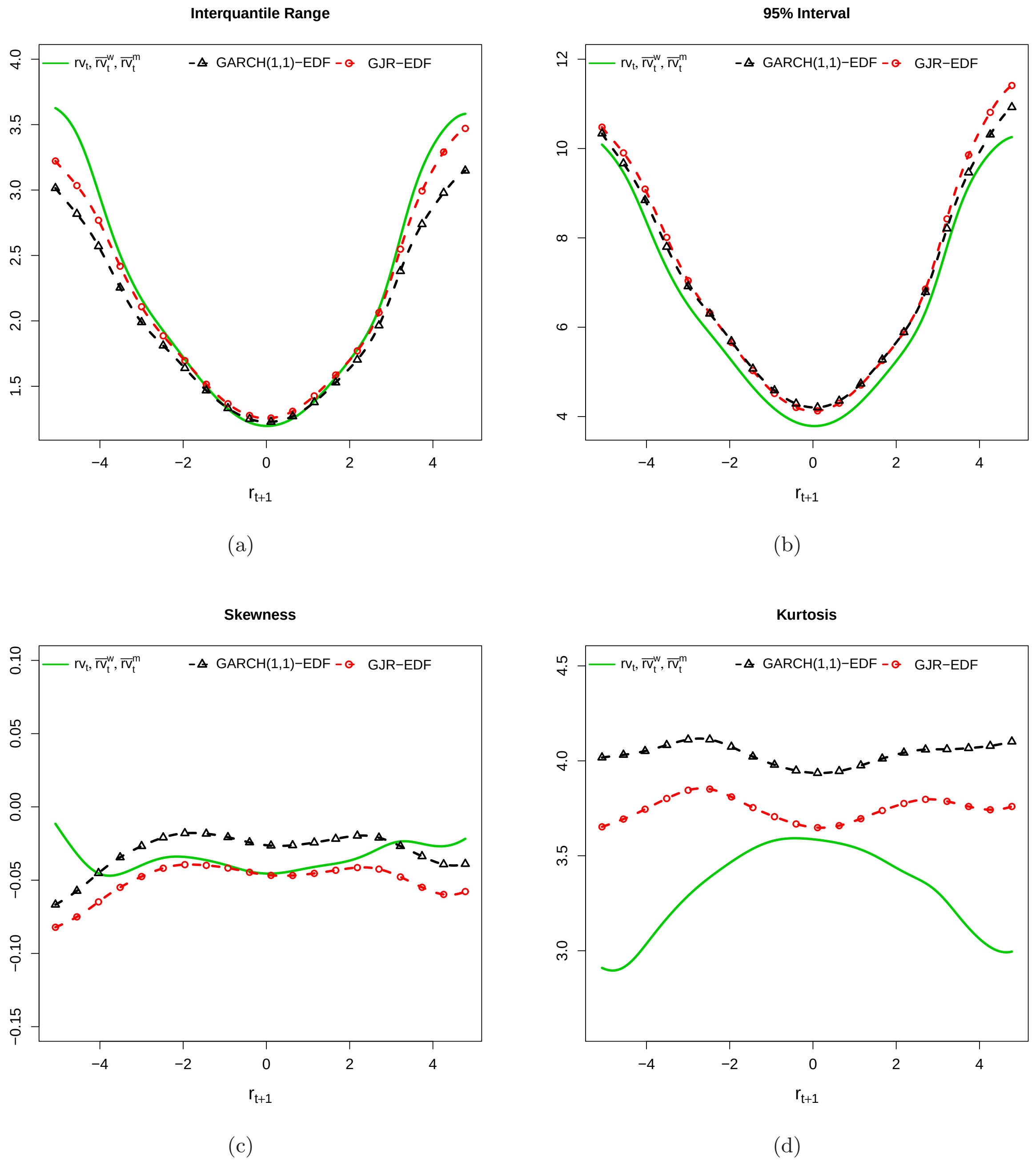



Figure 3: Kernel regression estimates of the four measures ($IQR_{t,1}$, 95% interval forecast, $SK_{t,1}$ and $K_{t,1}$) on the return realization, $r_{t+1}$, for the GARCH(1,1)-EDF, GJR-EDF, and the quantile model with $X_t = [rv_t, \overline{rv}_t^w, \overline{rv}_t^m]$.

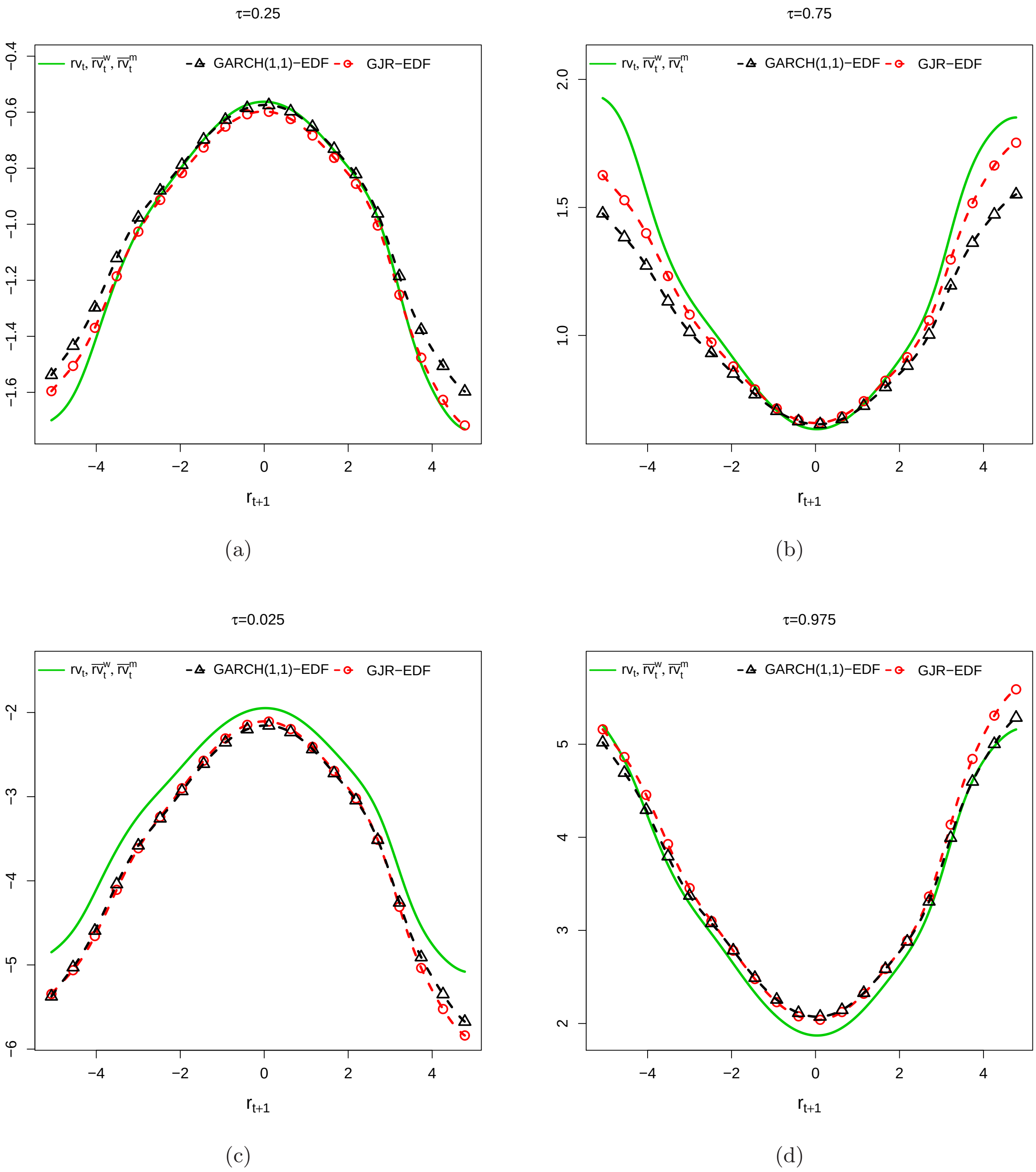



Figure 4: Kernel regression estimates of the quantile forecasts $q_{t,1}(\tau)$ (for $\tau$ equal to 0.025, 0.25, 0.75, and 0.975) on the return realization, $r_{t+1}$, for the GARCH(1,1)-EDF, GJR-EDF, and the quantile model with $X_t = [rv_t, \overline{rv}_t^w, \overline{rv}_t^m]$.

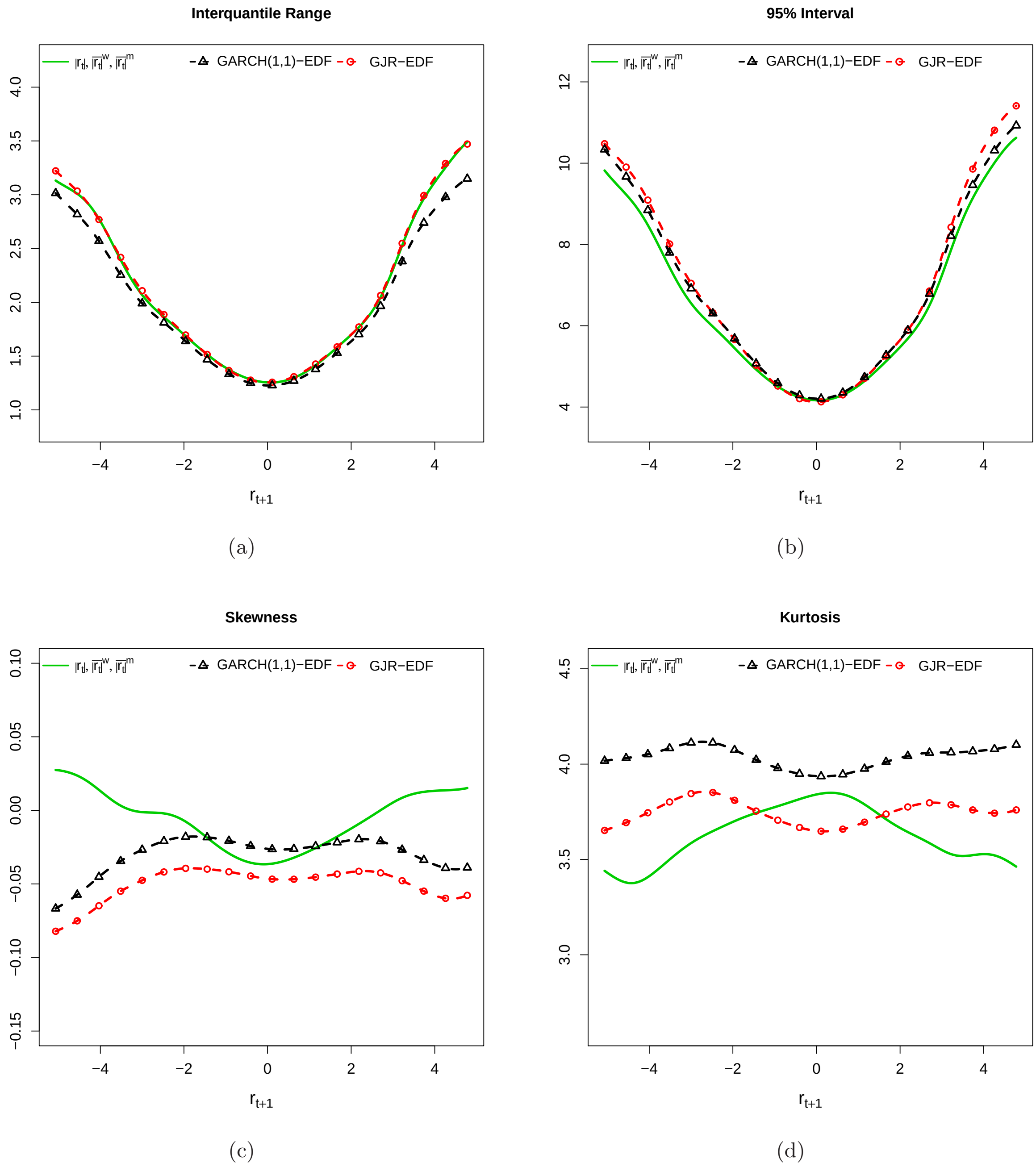



Figure 5: Kernel regression estimates of the four measures ($IQR_{t,1}$, 95% interval forecast, $SK_{t,1}$ and $K_{t,1}$) on the return realization, $r_{t+1}$, for the GARCH(1,1)-EDF, GJR-EDF, and the quantile model with $X_t = [|r_t|, |r_t|^w, |r_t|^m]$.

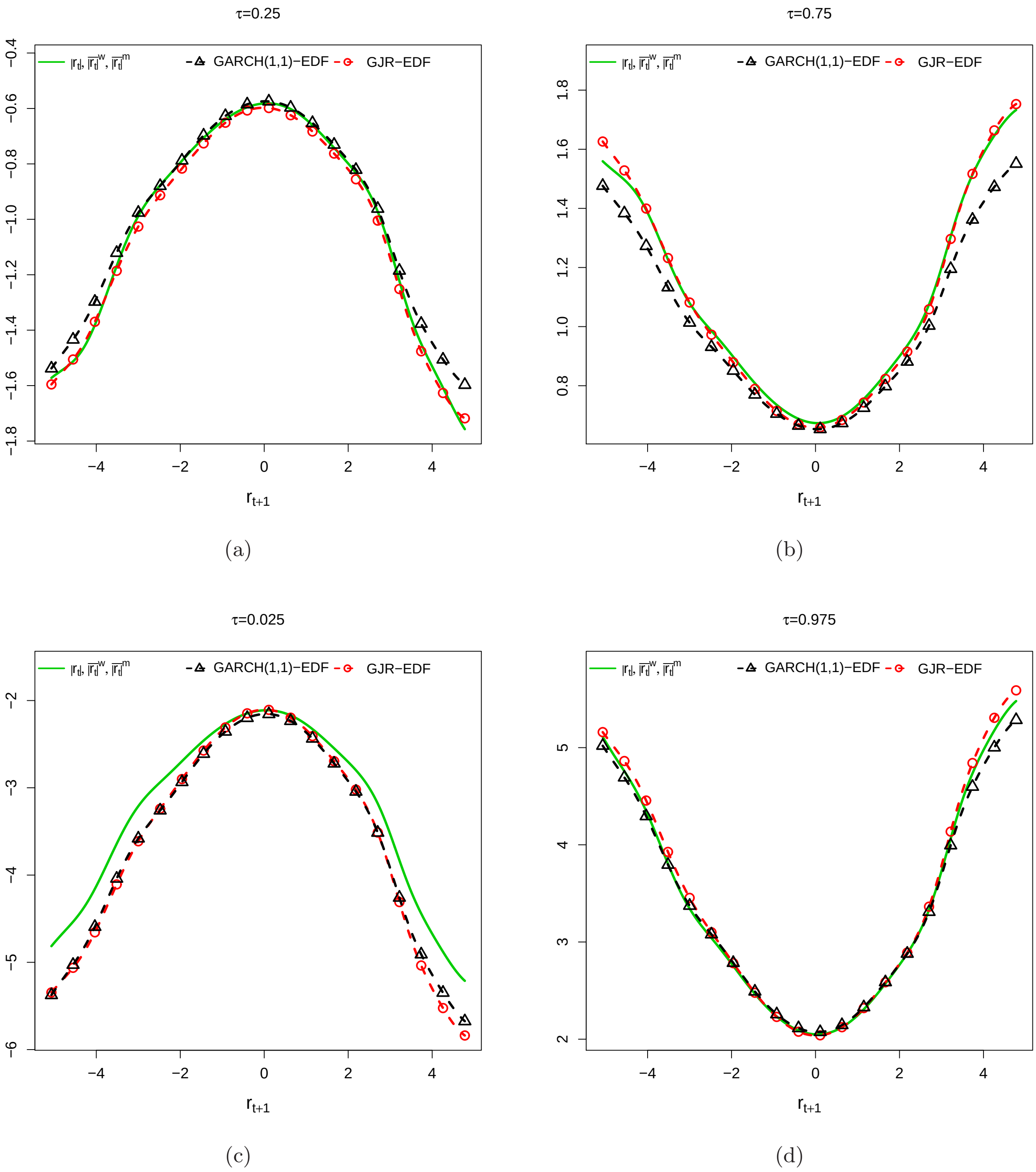



Figure 6: Kernel regression estimates of the quantile forecasts $q_{t,1}(\tau)$ (for $\tau$ equal to 0.025, 0.25, 0.75, and 0.975) on the return realization, $r_{t+1}$, for the GARCH(1,1)-EDF, GJR-EDF, and the quantile model with $X_t = [|r|_t, |r|_t^w, |r|_t^m]$.